MATH 320 - LINEAR ALGEBRA
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SPRING 2005, Quiz #2-Solutions by Turker Ozsari
You may not get credit unless you show all your work and write neatly!
You have 30 minutes.

(1) Prove that V is infinite dimensional if and only if there is a sequence $v_1, v_2, ...$ of vectors in V such that $(v_1, ..., v_n)$ is linearly independent for every positive integer n. (12.5 Points)

Proof. ($\Leftarrow$) Suppose there is a sequence $v_1, v_2, ...$ of vectors in V such that $(v_1, ..., v_n)$ is linearly independent for every positive integer n. Assume that V is finite dimensional, so there is a basis of V, $(u_1, ..., u_m)$ such that span$(u_1, ..., u_m) = V$. However, in a finite dimensional space the length of every linearly independent list of vectors in V is less than or equal to the length of every spanning list of vectors. Then we must have length$(v_1, ..., v_m, v_{m+1}) = m+1 \leq$length$(u_1, ..., u_m) = m$, contradiction. Hence, V is infinite dimensional.
($\Rightarrow$) Suppose V is infinite dimensional, i.e., there is not a spanning list in V. Then, $V \neq \{0\}$. Let $v_1 \neq 0$ be a vector in V. Since, B_1 :=span$(v_1) \neq V$, there is a $v_2 \in V - B_1$ such that (v_1, v_2) is linearly independent. Now, B_2 :=span$(v_1, v_2) \neq V$, there is a $v_3 \in V - B_2$ such that (v_1, v_2, v_3) is linearly independent, otherwise there are a_i, such that $a_1v_1 + a_2v_2 + a_3v_3 = 0$ and some $a_i \neq 0$. If, $a_3 \neq 0$, then $v_3 \in B_2$, contradiction and if $a_3 = 0$, (v_1, v_2) linearly dependent, contradiction. Going in this fashion, for $n \geq 4$, B_{n-1} =span$(v_1, ..., v_{n-1}) \neq V$, so there is $v_n \in V - B_{n-1}$ such that $(v_1, ..., v_n)$ is linearly independent, otherwise there are a_i, such that $\sum_{i=1}^{n} a_iv_i = 0$ and some $a_i \neq 0$. If, $a_n \neq 0$, then $v_n \in B_{n-1}$, contradiction and if $a_n = 0$, $(v_1, ..., v_{n-1})$ linearly dependent, contradiction. Hence, there is a sequence $v_1, v_2, ...$ of vectors in V such that $(v_1, ..., v_n)$ is linearly independent for every positive integer n.

(2) Which of the following are linear transformations on the real space $\mathbb{R}^2$? (7.5 Points)
(a) $T(x_1, x_2) = (1 + x_1, x_2)$
(b) $T(x_1, x_2) = (x_2, x_1)$
(c) $T(x_1, x_2) = (x_1^2, x_2)$
(d) $T(x_1, x_2) = (\sin x_1, x_2)$
(e) $T(x_1, x_2) = (x_1 - x_2, 0)$

Solution.
(a) Not a linear transformation, since $T(x_1+x_3, x_2+x_4) = (1+x_1+x_3, x_2+x_4) \neq T(x_1, x_2) + T(x_3, x_4) = (1 + x_1, x_2) + (1 + x_3, x_4) = (2 + x_1 + x_3, x_2 + x_4)$.
(b) Linear transformation. Since, $T(ax_1+bx_3, ax_2+bx_4) = (ax_2+bx_4, ax_1+bx_3) = a(x_2, x_1) + b(x_4, x_3) = aT(x_1, x_2) + bT(x_3, x_4)$.
(c) Not a linear transformation, since $T(1 + 1, 0 + 0) = (4, 0) \neq T(1, 0) + T(1, 0) = (1, 0) + (1, 0) = (2, 0)$.
(d) Not a linear transformation, since $T(\frac{\pi}{2} + \frac{\pi}{2}, 0 + 0) = T(\pi, 0) = (\sin \pi, 0) = (0, 0) \neq T(\frac{\pi}{2}, 0) + T(\frac{\pi}{2}, 0) = (1, 0) + (1, 0) = (2, 0)$.
(e) Linear transformation, since $T(ax_1, ax_2) = (ax_1 - ax_2, 0) = a(x_1 - x_2, 0) = aT(x_1, x_2)$ and $T(x_1+x_3, x_2+x_4) = (x_2+x_4-x_1-x_3, 0) = (x_2-x_1, 0)+(x_4-x_3, 0) = T(x_1, x_2) + T(x_3, x_4)$.