Math 320 - Fall 2020

Solutions to Quiz 1 Problems:

Problem 1: - Let $\vartheta : \mathbb{R} \to \mathbb{R}$ be the zero function,

$\forall x \in \mathbb{R}, \quad \vartheta(x) = 0 \quad \Rightarrow \quad \forall x \in [0, \infty), \ \vartheta(x) = 0$

$\Rightarrow \vartheta \in U$ ①

- $\forall g, h \in U, \ \exists a, b \in \mathbb{R}, \ \forall x \in [a, \infty), \ g(x) = 0$
  and $\forall x \in [b, \infty), \ h(x) = 0.$

  Let $c := a + b \Rightarrow \forall x \in [a+b, \infty),$

  $$x \geq a + b \Rightarrow \underset{\substack{\Downarrow \\ g(x) = 0}}{x \geq a} \text{ and } \underset{\substack{\Downarrow \\ h(x) = 0}}{x \geq b}$$

  $\Rightarrow (g + h)(x) = 0 \quad \text{for} \quad x \in [a+b, \infty)$

  $\Rightarrow g + h \in U$ ②

- $\forall \alpha \in \mathbb{R}, \ \forall g \in U, \ \exists a \in \mathbb{R}, \ \forall x \in [a, \infty), \ g(x) = 0$

  $\Rightarrow (\alpha g)(x) = \alpha g(x) = 0 \Rightarrow \alpha g \in U$ ③

①, ②, ③ $\Rightarrow U$ is a subspace of V. $\blacksquare$

Problem 2:

2.a) $\forall v \in Span(A+B)$, $\exists n \in \mathbb{Z}^+$, $\exists \alpha_1, \dots, \alpha_n \in \mathbb{F}$
$\exists a_1, \dots, a_n \in A$, $\exists b_1, \dots, b_n \in B$,

$$v = \alpha_1(a_1+b_1) + \alpha_2(a_2+b_2) + \dots + \alpha_n(a_n+b_n)$$

$$= \underbrace{\alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_n a_n}_{\in Span(A)} + \underbrace{\alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n}_{Span(B)}$$

$$\in Span(A) + Span(B)$$

$\Rightarrow \quad Span(A+B) \subseteq Span(A) + Span(B)$

so the answer is "Yes".

2.b) No, because in general if $a \in A$ and $b \in B$, $a + 2b \in Span(A) + Span(B)$ but $a+2b$ may not belong to $Span(A+B)$.
For example let $V = \mathbb{R}^2$, $a := (1,0)$,
$b := (0,1)$, $A := \{a\}$, $B := \{b\}$ $\Rightarrow$
$A + B = \{a+b\} = \{(1,1)\}$
$\Rightarrow Span(A+B) = \{\alpha(1,1) \mid \alpha \in \mathbb{R}\} = \{(\alpha,\alpha) \mid \alpha \in \mathbb{R}\}$
$Span(A) + Span(B) = \{\alpha(1,0) + \beta(0,1) \mid \alpha, \beta \in \mathbb{R}\}$
$= \mathbb{R}^2$

$\Rightarrow (1,2) \in Span(A) + Span(B)$ but
$(1,2) \notin Span(A+B)$

$\Leftrightarrow \quad Span(A) + Span(B) \not\subseteq Span(A+B)$.

<u>Problem 3:</u> $\mathbb{C}^2 = U_1 \oplus U_2 \iff \forall z \in \mathbb{C}^2$

$\exists!\, b \in U_1$ & $\exists!\, c \in U_2$, $z = b + c$ ①

$U_1 = \text{Span}(\{a\})$, $U_2 = \text{Span}(\{\bar{a}\})$

$\iff \exists!\, \beta \in \mathbb{C}$, $b = \beta a = \beta(\alpha_1, \alpha_2)$
$= (\beta\alpha_1, \beta\alpha_2)$ ②

$\iff \exists!\, \gamma \in \mathbb{C}$, $c = \gamma \bar{a} = \gamma(\bar{\alpha}_1, \bar{\alpha}_2)$
$= (\gamma\bar{\alpha}_1, \gamma\bar{\alpha}_2)$ ③

$z \in \mathbb{C}^2 \Rightarrow \exists\, z_1, z_2 \in \mathbb{C}$, $z = (z_1, z_2)$ ④

① - (4) $\Rightarrow$ $(z_1, z_2) = (\beta\alpha_1, \beta\alpha_2) + (\gamma\bar{\alpha}_1, \gamma\bar{\alpha}_2)$
$= (\beta\alpha_1 + \gamma\bar{\alpha}_1, \beta\alpha_2 + \gamma\bar{\alpha}_2)$

$$\Rightarrow \begin{cases} \beta\alpha_1 + \gamma\bar{\alpha}_1 = z_1 \\ \beta\alpha_2 + \gamma\bar{\alpha}_2 = z_2 \end{cases} \Rightarrow \begin{bmatrix} \alpha_1 & \bar{\alpha}_1 \\ \alpha_2 & \bar{\alpha}_2 \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

This show that $\mathbb{C}^2 = U_1 \oplus U_2$ if and only if this system of equations has a unique solution.

$$\iff \det \begin{bmatrix} \alpha_1 & \bar{\alpha}_1 \\ \alpha_2 & \bar{\alpha}_2 \end{bmatrix} \neq 0 \iff \alpha_1\bar{\alpha}_2 - \alpha_2\bar{\alpha}_1 \neq 0$$
$$\iff \alpha_1\bar{\alpha}_2 \neq \alpha_2\bar{\alpha}_1 = \overline{\alpha_1\bar{\alpha}_2}$$

$$\iff \boxed{\alpha_1\bar{\alpha}_2 \notin \mathbb{R}}.$$