Suggested Exercise Problems 7

Solve the following problems under the assumption that V is finite-dimensional.

Show that if V is a real inner-product space, then the set of self-adjoint operators on V is a subspace of $\mathcal{L}(V)$.

Show that if V is a complex inner-product space, then the set of self-adjoint operators on V is not a subspace of $\mathcal{L}(V)$.

Prove that if $T \in \mathcal{L}(V)$ is normal, then

$$\operatorname{range} T = \operatorname{range} T^*.$$

Prove that if $T \in \mathcal{L}(V)$ is normal, then

$$\operatorname{null} T^k = \operatorname{null} T \quad \text{and} \quad \operatorname{range} T^k = \operatorname{range} T$$

for every positive integer k.

Prove that the sum of any two positive operators on V is positive.

Suppose that T is a positive operator on V. Prove that T is invertible if and only if

$$\langle Tv, v \rangle > 0$$

for every $v \in V \setminus \{0\}$.

Prove or give a counterexample: if $S \in \mathcal{L}(V)$ and there exists an orthonormal basis $(e_1, \ldots, e_n)$ of V such that $\|Se_j\| = 1$ for each e_j, then S is an isometry.

Suppose $T \in \mathcal{L}(V)$ is self-adjoint, $\lambda \in \mathbf{F}$, and $\epsilon > 0$. Prove that if there exists $v \in V$ such that $\|v\| = 1$ and

$$\|Tv - \lambda v\| < \epsilon,$$

then T has an eigenvalue λ' such that $|\lambda - \lambda'| < \epsilon$.