Suggested Exercise Problems 6

Prove that

$$\Big(\sum_{j=1}^{n} a_j b_j\Big)^2 \leq \Big(\sum_{j=1}^{n} j a_j{}^2\Big)\Big(\sum_{j=1}^{n} \frac{b_j{}^2}{j}\Big)$$

for all real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$.

Prove that if V is a real inner-product space, then

$$\langle u, v \rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}$$

for all $u, v \in V$.

Prove that if V is a complex inner-product space, then

$$\langle u, v \rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + \|u+iv\|^2 i - \|u-iv\|^2 i}{4}$$

for all $u, v \in V$.

Suppose $(e_1, \ldots, e_m)$ is an orthonormal list of vectors in V. Let $v \in V$. Prove that

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2$$

if and only if $v \in \operatorname{span}(e_1, \ldots, e_m)$.

Prove that if $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and

$$\|Pv\| \leq \|v\|$$

for every $v \in V$, then P is an orthogonal projection.

Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U is invariant under T if and only if $P_U T P_U = T P_U$.

Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U and $U^{\perp}$ are both invariant under T if and only if $P_U T = T P_U$.