Suggested Exercise Problems 3

Suppose that V is finite dimensional. Prove that any linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

Suppose that V is finite dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap \operatorname{null} T = \{0\}$ and $\operatorname{range} T = \{Tu : u \in U\}$.

Suppose that V and W are both finite dimensional. Prove that there exists a surjective linear map from V onto W if and only if $\dim W \leq \dim V$.

Suppose that V and W are finite dimensional and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$ if and only if $\dim U \geq \dim V - \dim W$.

Suppose that U and V are finite-dimensional vector spaces and that $S \in \mathcal{L}(V, W)$, $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T.$$

Prove that every linear map from $\operatorname{Mat}(n, 1, \mathbf{F})$ to $\operatorname{Mat}(m, 1, \mathbf{F})$ is given by a matrix multiplication. In other words, prove that if $T \in \mathcal{L}(\operatorname{Mat}(n, 1, \mathbf{F}), \operatorname{Mat}(m, 1, \mathbf{F}))$, then there exists an m-by-n matrix A such that $TB = AB$ for every $B \in \operatorname{Mat}(n, 1, \mathbf{F})$.

Suppose that V is finite dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$.

Suppose that V is finite dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.