Math 303: Remedial Exam

Spring 2020

- This is a closed-book online oral exam to be conducted on Skype.

- There is a specific time for solving each problem, by the end of which you must show your solution to the camera and send its photo or scanned copy to the following email address: **amostafazadeh@ku.edu.tr**.

- Write your name and student ID number in the solution page to each problem and sign. include a handwritten copy of the following honor code on the last solution page you submit and sign your name. "**I hereby certify that I have completed this exam on my own without any help from anyone else. I have not used, accessed or received any information from any source in taking this exam. The effort in the exam thus belongs completely to me**"

- To receive proper credit you should give detailed explanations and avoid making ambiguous statements and using illegible handwriting.

- In you response to exam problems you may use any result that is proven in class without proof, unless you are asked to produce the proof.

Problem 1 (**25 minutes**) Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a twice differentiable function, R be a closed region in $\mathbb{R}^2$ with a counterclockwise oriented smooth boundary ∂R, $\hat{n}$ be the unit normal outward vector to ∂R, and $ds := \sqrt{dx^2 + dy^2}$ be the line element of ∂R.

1.a (5 pts) Show that $\hat{n}\, ds = (dy, -dx)$.

1.b (10 pts) Use Green's theorem to show that the surface integral over R of the Laplacian of ϕ can be expressed as the line integral over ∂R of the directional derivative $D_{\hat{n}}$ of ϕ along $\hat{n}$, i.e.,

$$\iint_R \nabla^2\phi \; dx\, dy = \oint_{\partial R} D_{\hat{n}}\phi \; ds.$$

Problem 2 (**25 minutes, 20 pts**) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, and $u : \mathbb{R}^2 \to \mathbb{R}$, $v : \mathbb{R}^2 \to \mathbb{R}$, and $g : \mathbb{R}^2 \to \mathbb{R}$ be the functions defined by

$$u(x,y) := \mathrm{Re}[f(x+iy)], \quad v(x,y) := \mathrm{Im}[f(x+iy)], \quad g(x,y) := |f(x+iy)|.$$

Show that if (x_0, y_0) is a stationary point of g and $g(x_0, y_0) \neq 0$, then (x_0, y_0) is a stationary point of both u and v.

Problem 3 (**30 minutes**, 25 pts) Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(\pi x)}{x(x^2+1)^2}\, dx$.

Problem 4 (**25 minutes**) Let $\theta(x) := \begin{cases} 0 \text{ for } x < 0 \\ 1 \text{ for } x \geq 0 \end{cases}$, $f(x) := e^{|x|}\theta(1-x^2)$, and $g(x) := e^{-|x|} f'(x)$.

4.a (10 pts) Express $g(x)$ in terms of the step function θ and the Dirac delta function δ. Simplify your response as much as possible.

4.b (10 pts) Let $\tilde{g}(k)$ be the Fourier transform of $g(x)$. Find an explicit expression for $\dfrac{\tilde{g}(k)}{\tilde{g}(\pi)}$ and simplify it as much as possible.

Problem 5 (**25 minutes**, 20 pts) Let $u : \mathbb{R}^2 \to \mathbb{C}$ be a function and $\tilde{u}(k,t)$ be the Fourier-transform of $u(x,t)$ for each $t \in \mathbb{R}$, i.e.,

$$\tilde{u}(k,t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-ikx} dx.$$

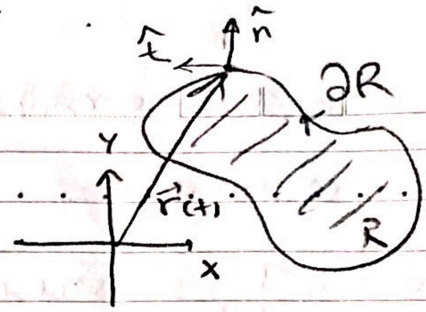
Use the following information to determine $u(x,t)$ for all $(x,t) \in \mathbb{R}^2$.

(i) There is a function $f : \mathbb{R} \to \mathbb{C}$ such that $\tilde{u}(k,t) = f(k)\, e^{-k\, t^2}$ for all $k \in \mathbb{R}$ and $t \in \mathbb{R}$.

(ii) $u(x,0) = \sin(x)$ for all $x \in \mathbb{R}$.

Solution to Remedial Exam

Math 203 - Spring 2020



Problem 1.a) $\vec{r} = (x, y)$

$\hat{t} := \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$

$\hat{n} = \hat{t} \times \hat{k}$

$\Rightarrow \hat{n}\, ds = \hat{n}\, |\vec{r}'(t)|\, dt = \left(\dfrac{\vec{r}'(t)}{|\vec{r}'(t)|} \times \hat{k}\right) |\vec{r}'(t)|\, dt$

$= \vec{r}'(t) \times \hat{k}\, dt = d\vec{r} \times \hat{k}$

$= \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ dx & dy & 0 \\ 0 & 0 & 1 \end{bmatrix} = \hat{i}(dy) - \hat{j}(dx)$

$= dy\,\hat{i} - dx\,\hat{j} = (dy, -dx)$

Problem 1.b) $\nabla^2 \phi = \vec{\nabla} \cdot \vec{\nabla} \phi = \partial_x(\partial_x \phi) + \partial_y(\partial_y \phi)$

Green's thm: $\iint_R (\partial_1 F_2 - \partial_2 F_1)\, dx\, dy = \oint_{\partial R} \vec{F} \cdot d\vec{r}$

$\partial_1 F_2 = \partial_x F_2 = \partial_x(\partial_x \phi) \Rightarrow$ let $F_2 = \partial_x \phi$

$-\partial_2 F_1 = -\partial_y F_1 = \partial_y(\partial_y \phi) \Rightarrow$ " $F_1 := -\partial_y \phi$

$\Rightarrow \iint_R \nabla^2 \phi\, dx\, dy = \oint_{\partial R} [(-\partial_y \phi)\, dx + \partial_x \phi\, dy]$

$= \oint_{\partial R} \vec{\nabla}\phi \cdot (dy, -dx)$

$= \oint_{\partial R} \vec{\nabla}\phi \cdot \hat{n}\, ds \qquad (D_{\hat{n}}\phi = \hat{n} \cdot \vec{\nabla}\phi)$

$= \oint_{\partial R} D_{\hat{n}}\phi\, ds.$ ▨

Problem 2 $\quad g(x,y) = |f(x+iy)| = [u(x,y)^2 + v(x,y)^2]^{1/2}$

$\vec{\nabla} g(x_0,y_0) = 0$

$$\vec{\nabla} g = \frac{1}{2g}(2u\vec{\nabla}u + 2v\vec{\nabla}v) = \frac{u\vec{\nabla}u + v\vec{\nabla}v}{g}$$

$$\vec{\nabla} v = (\partial_x v, \partial_y v) = (-\partial_y u, \partial_x u)$$

$$\vec{\nabla} g = 0 \Rightarrow u\vec{\nabla}u + v\vec{\nabla}v = 0$$

$$\Rightarrow u(\partial_x u, \partial_y u) + v(-\partial_y u, \partial_x u) = 0$$

$$\Rightarrow \begin{cases} u\partial_x u = v\partial_y u \\ u\partial_y u = -v\partial_x u \end{cases}$$

$$\Rightarrow u^2\partial_x u = uv\partial_y u = -v^2\partial_x u$$

$$\Rightarrow (u^2+v^2)\partial_x u = 0 \quad \& \quad g(x_0,y_0) \neq 0$$

$$\Rightarrow \boxed{\partial_x u(x_0,y_0) = 0} \Rightarrow \boxed{\partial_y v(x_0,y_0) = \partial_x u(x_0,y_0) = 0}$$

Also $v^2\partial_y u = v(u\partial_x u) = u(v\partial_x u) = -u^2\partial_y u$

$$\Rightarrow (v^2+u^2)\partial_y u = 0 \Rightarrow \boxed{\partial_y u(x_0,y_0) = 0}$$

$$\Rightarrow \boxed{\partial_x v(x_0,y_0) = -\partial_y u(x_0,y_0) = 0}$$

$$\Rightarrow \vec{\nabla} u(x_0,y_0) = \vec{0} \quad \& \quad \vec{\nabla} v(x_0,y_0) = \vec{0} \Rightarrow$$

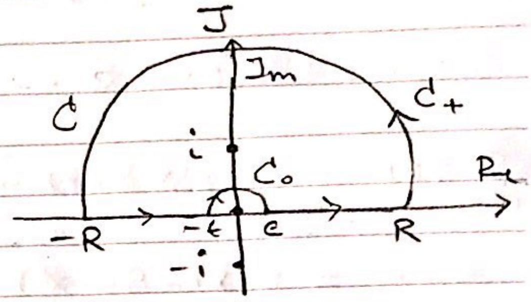
(x_0,y_0) is a stationary point of both u & v.

GIPTA

Problem 3: $I = \int_{-\infty}^{\infty} \frac{\sin(\pi x)}{x(x^2+1)^2}\,dx = \mathrm{Im}\Big(\text{P.V.} \underbrace{\int_{-\infty}^{\infty} \frac{e^{i\pi x}}{x(x^2+1)^2}\,dx}_{J}\Big)$

$$J = \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \left[\int_{-R}^{-\epsilon} f(x)\,dx + \int_{\epsilon}^{R} f(x)\,dx\right]$$

when $f(z) := \dfrac{e^{i\pi z}}{z(z^2+1)^2}$



$$\lim_{\substack{R\to\infty \\ \epsilon\to 0}} \oint_C f(z)\,dz = J + \int_{C_0} f(z)\,dz + \overbrace{\int_{C_+} f(z)\,dz}^{\to 0}$$

for $z \in C_0$, $z = \epsilon e^{i\theta}$ $\quad \theta \in [0,\pi]$

$dz = i\epsilon e^{i\theta}\,d\theta$

$$\int_{C_0} f(z)\,dz = \int_{\pi}^{0} f(\epsilon e^{i\theta})\, i\epsilon e^{i\theta}\,d\theta$$

$$\lim_{\epsilon\to 0} \int_{C_0} f(z)\,dz = \lim_{\epsilon\to 0} \int_{\pi}^{0} \frac{e^{i\pi\epsilon e^{i\theta}}}{\epsilon e^{i\theta}(\epsilon^2 e^{2i\theta}+1)^2}\, i\epsilon e^{i\theta}\,d\theta$$

$$= -i\int_0^{\pi} d\theta = -i\pi.$$

$$\lim_{\substack{R\to\infty \\ \epsilon\to 0}} \oint_C f(z)\,dz = 2\pi i\, \mathrm{Res}(i)$$

$f(z) = \dfrac{e^{i\pi z}}{z(z+i)^2(z-i)^2}$ $\qquad \lim_{z\to i} (z-i)^2 f(z) = \dfrac{e^{-\pi}}{i(2i)^2} \in \mathbb{C}\setminus\{0\}$

$\Rightarrow$ i is a double pole.

$g(z) := \dfrac{e^{i\pi z}}{z(z+i)^2} \Rightarrow \mathrm{Res}(i) = g'(i)$

$$g'(z) = \left[\frac{i\pi}{z(z+i)^2} - \frac{(z+i)^2 + 2z(z+i)}{z^2(z+i)^4}\right] e^{i\pi z}$$

$$g'(i) = \left[\frac{i\pi}{i(2i)^2} - \frac{(2i)^2 + 2i(2i)}{(i)^2(2i)^4}\right] e^{-\pi} = \left(-\frac{\pi}{4} - \frac{-4-4}{-16}\right) e^{-\pi}$$

$$= -\left(\frac{\pi}{4} + \frac{1}{2}\right) e^{-\pi} \Rightarrow \mathrm{Res}(i) = -\left(\frac{\pi}{4}+\frac{1}{2}\right) e^{-\pi}$$

$$\Rightarrow -2\pi i\left(\frac{\pi}{4}+\frac{1}{2}\right) e^{-\pi} = J - i\pi \Rightarrow J = i\pi\left[1 - \left(\frac{\pi}{2}+1\right) e^{-\pi}\right]$$

$$\Rightarrow I = \pi\left[1 - \left(\frac{\pi}{2}+1\right) e^{-\pi}\right]$$

GIPTA

Problem 4: $f(x) = e^{|x|}\, \theta(1-x^2)$

4.a) $f'(x) = \frac{d}{dx}\left[e^{|x|}\theta(1-x^2)\right] = e^{|x|}\theta(1-x^2)\, \frac{d}{dx}|x|$

$\qquad + e^{|x|} \frac{d}{dx}\theta(1-x^2)$

$\frac{d}{dx}|x| = \frac{d}{dx}\left[x\theta(x) - x\theta(-x)\right] = \theta(x) - \theta(-x) + x\delta(x) - x[-\delta(-x)]$

$= \theta(x) - \theta(-x)$

$\frac{d}{dx}\theta(1-x^2) = -2x\, \delta(1-x^2)$

$h(x) = 1-x^2 \Rightarrow h(x) = 0 \Rightarrow x = \pm 1, \quad h'(x) = -2x$

$\delta(1-x^2) = \frac{\delta(x-1)}{|h'(1)|} + \frac{\delta(x+1)}{|h'(-1)|} = \frac{1}{2}\left[\delta(x-1) + \delta(x+1)\right]$

$\Rightarrow g(x) = e^{-|x|} f'(x) = \theta(1-x^2)\left[\theta(x) - \theta(-x)\right]$

$\qquad + (-2x)\frac{1}{2}\left[\delta(x-1) + \delta(x+1)\right]$

$= \theta(1-x^2)\left[\theta(x) - \theta(-x)\right] - x\left[\delta(x-1) + \delta(x+1)\right]$

$= \theta(1-x^2)\left[\theta(x) - \theta(-x)\right] - \delta(x-1) + \delta(x+1)$

4.b) $\tilde{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} g(x)\, dx$

$= \frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{1}[\theta(x) - \theta(-x)]e^{-ikx}dx - e^{-ik} + e^{ik}\right]$

$= \frac{1}{\sqrt{2\pi}}\left[\underbrace{\int_{-1}^{0} e^{-ikx}dx}_{\frac{1-e^{ik}}{-ik}} - \underbrace{\int_{0}^{1} e^{-ikx}dx}_{\frac{e^{-ik}-1}{-ik}} + e^{ik} - e^{-ik}\right]$

$= \frac{1}{\sqrt{2\pi}}\left[-\left(\frac{1}{ik} + \frac{1}{ik}\right) + \frac{1}{ik}\left(e^{ik} + e^{-ik}\right) + e^{ik} - e^{-ik}\right]$

$= \frac{1}{\sqrt{2\pi}}\left(\frac{2i}{k} - \frac{2i\cos k}{k} + 2i\sin k\right) = i\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos k}{k} + \sin k\right)$

$\tilde{g}(\pi) = i\sqrt{\frac{2}{\pi}}\left(\frac{2}{k}\right) \hookrightarrow \frac{\tilde{g}(k)}{\tilde{g}(\pi)} = \frac{1}{2}(1 - \cos k + k\sin k)$.

Problem 5 : $\tilde{u}(k,t) = f(k)\, e^{-kt^2}$, $u(x,0) = \sin x$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}\, \tilde{u}(k,t)\, dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}\, f(k)\, e^{-kt^2}\, dk$$

$$\sin x = u(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(k)\, dk$$

$$\Rightarrow f(k) = \mathcal{F}\{\sin x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{ix} - e^{-ix}}{2i}\, e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{2i} \int_{-\infty}^{\infty} [e^{-i(k-1)x} - e^{-i(k+1)x}]\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{2i}\, 2\pi [\delta(k-1) - \delta(k+1)]$$

$$= \frac{\sqrt{2\pi}}{2i} [\delta(k-1) - \delta(k+1)]$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{2\pi}}{2i} \int_{-\infty}^{\infty} e^{ikx} [\delta(k-1) - \delta(k+1)]\, e^{-kt^2}\, dt$$

$$= \frac{1}{2i} (e^{ix-t^2} - e^{-ix+t^2})$$

$$= \frac{1}{2i} [e^{i(x+it^2)} - e^{-i(x+it^2)}]$$

$$= \sin(x+it^2)$$