Math 303: Final Exam (Part 1)

May 31, 2020

- This exam consists of five separate 30 minutes-long parts.

- Solve the problem(s) assigned to you. Otherwise you will not be getting any credit.

- Write your name and student ID number in the solution page to each problem, include a handwritten copy of the following honor code, and place your signature underneath. "**I hereby certify that I have completed this exam on my own without any help from anyone else. I understand that the only sources of authorized information in this open-book exam are (i) the course textbook and (ii) the material that is posted at Blackboard for this class, available to all other students. I have not used, accessed or received any information from any other unauthorized source in taking this exam. The effort in the exam thus belongs completely to me.**"

- Take a photo of the solution page(s) for each problem and upload it to Blackboard as a single electronic file before the due time indicated in the assignment. In addition, send the same file via email to **amostafazadeh@ku.edu.tr** before or within the last three minutes after the end of the time given for each problem. Late submissions of the solution to the problems will not be taken into consideration.

Problem 1 for students with ID numbers: 40600, 49960, 54212, 60333, 64365, 64821

1.a (5 pts) Let $r : \mathbb{R}^3 \to \mathbb{R}$ be defined by $r(x_1, x_2, x_3) := \sqrt{\sum_{i=1}^3 x_i^2}$. By performing explicit calculations, express $\boldsymbol{\nabla} \cdot \left(\dfrac{\boldsymbol{\nabla} r}{r}\right)$ as a function of r wherever it exists.

1.b (10 pts) Given arbitrary twice differentiable functions $\phi : \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$, use the properties of the Kronecker delta and Levi Civita epsilon symbols to find functions $B_{ijk}, C_{ij} : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{A} \cdot \big[\boldsymbol{\nabla} \times \big(\phi \boldsymbol{\nabla} \phi \times \mathbf{A}\big)\big] = \displaystyle\sum_{i,j=1}^{3} A_i L_{ij} A_j$ and $L_{ij} := \displaystyle\sum_{k=1}^{3} B_{ijk} \frac{\partial^k}{\partial x^k} + C_{ij}$.

Problem 1 for students with ID numbers: 50127, 64558, 64869, 54238, 60581, 69720

1.a (5 pts) Let $r : \mathbb{R}^3 \to \mathbb{R}$ be defined by $r(x_1, x_2, x_3) := \sqrt{\sum_{i=1}^3 x_i^2}$. By performing explicit calculations, express $\boldsymbol{\nabla} \cdot \left(r \boldsymbol{\nabla} \dfrac{1}{r}\right)$ as a function of r wherever it exists.

1.b (10 pts) Given arbitrary twice differentiable functions $\phi : \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$, use the properties of the Kronecker delta and Levi Civita epsilon symbols to find functions $B_{ijk}, C_{ij} : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{A} \cdot \big\{\boldsymbol{\nabla} \phi \times \big[\boldsymbol{\nabla} \times \big(\phi \mathbf{A}\big)\big]\big\} = \displaystyle\sum_{i,j=1}^{3} A_i L_{ij} A_j$ and $L_{ij} := \displaystyle\sum_{k=1}^{3} B_{ijk} \frac{\partial^k}{\partial x^k} + C_{ij}$.

Problem 1 for students with ID numbers: 49642, 64842, 69287, 40557, 53932

1.a (5 pts) Let $r : \mathbb{R}^3 \to \mathbb{R}$ be defined by $r(x_1, x_2, x_3) := \sqrt{\sum_{i=1}^3 x_i^2}$. By performing explicit calculations, express $\dfrac{\nabla^2 r}{r}$ as a function of r wherever it exists.

1.b (10 pts) Given arbitrary twice differentiable functions $\phi : \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$, use the properties of the Kronecker delta and Levi Civita epsilon symbols to find functions $B_{ijk}, C_{ij} : \mathbb{R}^3 \to \mathbb{R}$ such that $\boldsymbol{\nabla} \phi \cdot \big\{\mathbf{A} \times \big[\boldsymbol{\nabla} \times \big(\phi \mathbf{A}\big)\big]\big\} = \displaystyle\sum_{i,j=1}^{3} A_i L_{ij} A_j$ and $L_{ij} := \displaystyle\sum_{k=1}^{3} B_{ijk} \frac{\partial^k}{\partial x^k} + C_{ij}$.

Problem 1 / Group 1

1.a $r = \left(\sum_i x_i^2\right)^{1/2}$

$$\partial_j r = \frac{1}{2}\left(\sum_i x_i^2\right)^{-1/2} \sum_i 2x_i \delta_{ij} = \frac{x_j}{r} \Rightarrow \vec{\nabla} r = \frac{\vec{r}}{r}$$

$$\vec{\nabla}\cdot\left(\frac{\vec{\nabla} r}{r}\right) = \sum_j \partial_j\left(\frac{x_j}{r^2}\right) = \frac{1}{r^2}\sum_{j=1}^{3}\partial_j x_j + \sum_{j=1}^{3} x_j\, \partial_j \left(\sum_i x_i^2\right)^{-1}$$

$$= \frac{3}{r^2} + \sum_{j=1}^{3} x_j \underbrace{\left[-\left(\sum_i x_i^2\right)^{-2}(2x_i\delta_{ij})\right]}_{-\frac{2x_j}{r^4}}$$

$$= \frac{3}{r^2} - \frac{2}{r^2} = \frac{1}{r^2}$$

1.b $\vec{A}\cdot[\vec{\nabla}\times(\phi\vec{\nabla}\phi\times\vec{A})] = \frac{1}{2}\vec{A}\cdot[\vec{\nabla}\times(\vec{\nabla}\phi^2\times\vec{A})]$

$$= \frac{1}{2}\sum_{i=1}^{3} A_i \sum_{j,k=1}^{3}\epsilon_{ijk}\,\partial_j \sum_{l,m=1}^{3}\epsilon_{klm}(\partial_l\phi^2)A_m$$

$$= \frac{1}{2}\sum_{i,j,k,l,m=1}^{3} A_i\,\epsilon_{kij}\,\epsilon_{klm}\,\partial_j[(\partial_l\phi^2)A_m]$$

$$= \frac{1}{2}\sum_{\substack{i,j,\\ l,m=1}}^{3}\left[A_i(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})[(\partial_j\partial_l\phi^2)A_m + (\partial_l\phi^2)\partial_j A_m]\right]$$

$$= \frac{1}{2}\sum_{i,j=1}^{3} A_i\left[(\partial_j\partial_i\phi^2)A_j + (\partial_i\phi^2)\partial_j A_j - (\partial_j\partial_j\phi^2)A_i - (\partial_j\phi^2)\partial_j A_i\right]$$

$$= \frac{1}{2}\Big[\sum_{i,j=1}^{3} A_i[\partial_i\partial_j\phi^2 + (\partial_i\phi^2)\partial_j]A_j - \left(\sum_{i=1}^{3}A_i^2\right)\underbrace{\left(\sum_{j=1}^{3}\partial_j^2\phi^2\right)}_{\nabla^2\phi^2} - \sum_{i=1}^{3}A_i\underbrace{\left(\sum_{j=1}^{3}(\partial_j\phi^2)\partial_j\right)}_{\sum_{k=1}^{3}(\partial_k\phi^2)\partial_k}A_i\Big]$$

$$= \sum_{i,j=1}^{3} A_i L_{ij} A_j \Rightarrow$$

$$L_{ij} = \frac{1}{2}\Big[\partial_i\partial_j\phi^2 + \underbrace{(\partial_i\phi^2)\partial_j}_{\sum_{k=1}^{3}\delta_{jk}(\partial_i\phi^2)\partial_k} - \nabla^2\phi^2\,\delta_{ij} - \delta_{ij}\sum_{k=1}^{3}(\partial_k\phi^2)\partial_k\Big]$$

$$\Rightarrow \quad L_{ij} = \sum_{k=1}^{3} \frac{1}{2}\left(-\delta_{ij}\,\partial_k \phi^2 + \delta_{jk}\,\partial_i \phi^2\right)\partial_k \overset{(2)}{}$$

$$+ \frac{1}{2}\left(\partial_i\partial_j \phi^2 - \nabla^2 \phi^2 \delta_{ij}\right)$$

$$\Rightarrow \quad B_{ijk} = \frac{1}{2}\left(\delta_{jk}\,\partial_i \phi^2 - \delta_{ij}\,\partial_k \phi^2\right)$$

$$C_{ij} = \frac{1}{2}\left(\partial_i\partial_j \phi^2 - \nabla^2\phi^2 \delta_{ij}\right)$$

Problem 1 / Group 2

1.a

$$r^{-1} = \left(\sum_{i=1}^{3} x_i^2\right)^{-1/2}$$

$$\partial_j r^{-1} = -\frac{1}{2}\left(\sum_{i=1}^{3} x_i^2\right)^{-3/2} \sum_{i=1}^{3} 2x_i \delta_{ij} = -\frac{x_j}{r^3}$$

$$\vec{\nabla}\cdot\left(r\vec{\nabla}\frac{1}{r}\right) = \sum_{j=1}^{3} \partial_j\left(-\frac{x_j}{r^2}\right) = -\frac{1}{r^2}\sum_{j=1}^{3}\partial_j x_j - \sum_{j=1}^{3} x_j\,\partial_j (r^{-2})$$

$$= -\frac{3}{r^2} - \sum_{j=1}^{3} x_j \underbrace{\partial_j \left(\sum_{i=1}^{3} x_i^2\right)^{-1}}_{-\left(\sum_{i=1}^{3} x_i^2\right)^{-2} \sum_{i=1}^{3} 2x_i\delta_{ij}}$$

$$= -\frac{3}{r^2} + 2\sum_{j=1}^{3}\frac{x_j^2}{r^4} = -\frac{1}{r^2}$$

1.b

$$\vec{A}\cdot\left\{\vec{\nabla}\phi \times \left[\vec{\nabla}\times(\phi\vec{A})\right]\right\}$$

$$= \sum_{i=1}^{3} A_i \sum_{j,k=1}^{3} \epsilon_{ijk}(\partial_j\phi) \sum_{l,m=1}^{3} \epsilon_{klm}\,\partial_l(\phi A_m)$$

$$= \sum_{i,j,k,l,m=1}^{3} A_i\, \epsilon_{kij}\,\epsilon_{klm}\,(\partial_j\phi)\,\partial_l(\phi A_m)$$

$$= \sum_{i,j,l,m=1}^{3} A_i(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(\partial_j\phi)\left[(\partial_l\phi)A_m + \phi\,\partial_l A_m\right]$$

$$= \sum_{i,j=1}^{3} A_i(\partial_j\phi)\left[(\partial_i\phi)A_j + \phi\,\partial_i A_j - (\partial_j\phi)A_i - \phi\,\partial_j A_i\right]$$

$$= \sum_{i,j=1}^{3} A_i \left[(\partial_i \phi)(\partial_j \phi) + \phi (\partial_j \phi) \sum_{k=1}^{3} \delta_{ik} \partial_k \right] A_j$$

$$- \left(\sum_{i=1}^{3} A_i^2\right) \underbrace{\left[\sum_{j=1}^{3} (\partial_j \phi)^2\right]}_{|\vec{\nabla}\phi|^2} - \phi \sum_{i=1}^{3} A_i \underbrace{\left[\sum_{j=1}^{3} (\partial_j \phi) \partial_j\right]}_{\sum_{k=1}^{3} (\partial_k \phi)\partial_k} A_i$$

$$= \sum_{i,j=1}^{3} A_i L_{ij} A_j$$

$$\Rightarrow \quad L_{ij} = (\partial_i \phi)(\partial_j \phi) - |\vec{\nabla}\phi|^2 \delta_{ij} + \sum_{k=1}^{3} \phi \left[(\partial_j \phi)\delta_{ik} - (\partial_k \phi)\delta_{ij} \right] \partial_k$$

$$\Rightarrow \quad B_{ijk} = \phi \left[(\partial_j \phi)\delta_{ik} - (\partial_k \phi)\delta_{ij} \right]$$

$$= \frac{1}{2} \left[(\partial_j \phi^2)\delta_{ik} - (\partial_k \phi^2)\delta_{ij} \right]$$

$$\& \quad C_{ij} = (\partial_i \phi)(\partial_j \phi) - |\vec{\nabla}\phi|^2 \delta_{ij}$$

<u>Problem 1 / Group 3</u>

<u>1.a</u> $r = \left(\sum_{i=1}^{3} x_i^2\right)^{1/2} \Rightarrow \partial_j r = \frac{1}{2} \left(\sum_{i=1}^{3} x_i^2\right)^{-1/2} \sum_{i=1}^{3} 2 x_i \delta_{ij} = \frac{x_j}{r}$

$$\partial_j^2 r = \partial_j \left(\frac{x_j}{r}\right) = \frac{1}{r} + x_j \partial_j \left(\sum_{i=1} x_i^2\right)^{-1/2}$$

$$= \frac{1}{r} + x_j \left[-\frac{1}{2} \left(\sum_{i=1}^{3} x_i^2\right)^{-3/2} \sum_{i=1}^{3} 2 x_i \delta_{ij} \right]$$

$$= \frac{1}{r} - \frac{x_j^2}{r^3} \quad \Rightarrow \quad \nabla^2 r = \sum_{j=1}^{3} \partial_j^2 r = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

$$\Rightarrow \quad \frac{\nabla^2 r}{r} = \frac{2}{r^2}$$

1.b $\vec{\nabla}\phi \cdot \{\vec{A} \times [\vec{\nabla} \times (\phi \vec{A})]\}$

$$= \sum_{i=1}^{3} (\partial_i \phi) \sum_{j,k=1}^{3} \epsilon_{ijk} A_j \Big[\sum_{l,m=1}^{3} \epsilon_{klm} \partial_l (\phi A_m)\Big]$$

$$= \sum_{\substack{i,j,k,\\ l,m=1}}^{3} \epsilon_{kij} \epsilon_{klm} (\partial_i \phi) A_j \partial_l (\phi A_m)$$

$$= \sum_{\substack{i,j,\\ l,m=1}}^{3} (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(\partial_i \phi) A_j [(\partial_l \phi) A_m + \phi \partial_l A_m]$$

$$= \sum_{i,j=1}^{3} (\partial_i \phi) A_j [(\partial_i \phi) A_j + \phi \partial_i A_j - (\partial_j \phi) A_i - \phi \partial_j A_i]$$

$$= \underbrace{\Big[\sum_{i=1}^{3} (\partial_i \phi)^2\Big]}_{|\vec{\nabla}\phi|^2} \underbrace{\Big[\sum_{j=1}^{3} A_j^2\Big]}_{|\vec{A}|^2} + \sum_{i,j=1}^{3} \Big[A_j (\partial_i \phi) \partial_i A_j - A_i (\partial_i \phi)(\partial_j \phi) A_j - \phi A_j (\partial_i \phi) \partial_j A_i\Big]$$

$$= |\vec{\nabla}\phi|^2 |\vec{A}|^2 + \sum_{j,k=1}^{3} A_j (\partial_k \phi) \partial_k A_j - \sum_{i,j=1}^{3} A_i (\partial_i \phi)(\partial_j \phi) A_j - \sum_{i,j=1}^{3} \phi A_i (\partial_j \phi) \partial_i A_j$$

$$= \sum_{i,j=1}^{3} \Big\{ |\vec{\nabla}\phi|^2 A_i \delta_{ij} A_j + \delta_{ij} A_i \sum_{k=1}^{3} (\partial_k \phi) \partial_k A_j - A_i (\partial_i \phi)(\partial_j \phi) A_j - \phi A_i \sum_{k=1}^{3} (\partial_j \phi) \delta_{ik} \partial_k A_j \Big\}$$

$$= \sum_{i,j=1}^{3} A_i L_{ij} A_j \Rightarrow L_{ij} = \sum_{k=1}^{3} B_{ijk} \partial_k + C_{ij} \quad \text{with}$$

$$B_{ijk} = \delta_{ij} (\partial_k \phi) - \phi (\partial_j \phi) \delta_{ik}$$

$$\& \quad C_{ij} = |\vec{\nabla}\phi|^2 \delta_{ij} - (\partial_i \phi)(\partial_j \phi).$$

Math 303: Final Exam (Part 2)

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Problem 2 for students with ID numbers: 40600, 49960, 54212, 60333, 64365, 64821

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $\mathrm{Im}[f(0)] = 1$, and suppose that there is a function $q : \mathbb{R} \to \mathbb{R}$ such that for all $(x, y) \in \mathbb{R}^2$, $\mathrm{Re}[f(x + iy)] = (x \cos x + y \sin x)q(y) - x$ and $q(0) = -1$.

2.a (10 pts) Show that $q(y)$ satisfies $yq''(y) + 2q'(y) - (y + 2)q(y) = 0$.

2.b (10 pts) Find an explicit formula for $q(y)$.

2.c (Optional 10 bonus pts) Find an explicit formula for $f(z)$.

Problem 2 for students with ID numbers: 50127, 64558, 64869, 54238, 60581, 69720

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $\mathrm{Re}[f(0)] = -1$, and suppose that there is a function $q : \mathbb{R} \to \mathbb{R}$ such that for all $(x, y) \in \mathbb{R}^2$, $\mathrm{Im}[f(x + iy)] = (y \cos x - x \sin x)q(y) - y$ and $q(0) = 2$.

2.a (10 pt) Show that $q(y)$ satisfies $yq''(y) + 2q'(y) - (y + 2)q(y) = 0$.

2.b (10 pt) Find an explicit formula for $q(y)$.

2.c (Optional 10 bonus pts) Find an explicit formula for $f(z)$.

Problem 2 for students with ID numbers: 49642, 64842, 69287, 40557, 53932

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $\mathrm{Re}[f(0)] = 2$, and suppose that there is a function $q : \mathbb{R} \to \mathbb{R}$ such that for all $(x, y) \in \mathbb{R}^2$, $\mathrm{Im}[f(x + iy)] = (y \cos x - x \sin x)q(y) + y$ and $q(0) = -1$.

2.a (10 pt) Show that $q(y)$ satisfies $yq''(y) + 2q'(y) - (y + 2)q(y) = 0$.

2.b (10 pt) Find an explicit formula for $q(y)$.

2.c (Optional 10 bonus pts) Find an explicit formula for $f(z)$.

Problem 2
Group 1

2.a $\quad u(x,y) = (x\cos x + y\sin x)\, q(y) \qquad q(0) = -1$

$\nabla^2 u = 0$

$u_x = (\cos x - x\sin x + y\cos x)\, q$

$u_{xx} = (-\sin x - \sin x - x\cos x - y\sin x)\, q$

$u_y = (\sin x)\, q + (x\cos x + y\sin x)\, q'$

$u_{yy} = (\sin x)\, q' + (\sin x)\, q' + (x\cos x + y\sin x)\, q''$

$$\nabla^2 u = (-2\sin x - x\cos x - y\sin x)\, q + (2\sin x)\, q' + (x\cos x + y\sin x)\, q''$$

$\nabla^2 u = 0 \Rightarrow \sin x\,[(-2-y)q + 2q' + yq''] + x\cos x\,(-q + q'') = 0$

$\sin x$ & $x\cos x$ are linearly independent $\Rightarrow$

$yq'' + 2q' - (y+2)q = 0 \qquad (1)$

& $q'' - q = 0 \qquad (2)$

2.b $\quad (1)$ & $(2) \Rightarrow yq + 2q' - (y+2)q = 0$

$\Rightarrow 2q' - 2q = 0 \Rightarrow q' = q$

$\Rightarrow q(y) = ce^y$ for some $c \in \mathbb{R}$

$q(0) = -1 \Rightarrow c = -1 \Rightarrow \boxed{q(y) = -e^y}$

2.c $\quad u = -(x\cos x + y\sin x)e^y$

$$u_x = -(\cos x - x\sin x + y\cos x)e^y$$

$$v_y = u_x \Rightarrow v = \int u_x \, dy$$

$$v = -\int (\cos x - x\sin x + y\cos x)e^y \, dy + h(x)$$

$$= -(\cos x - x\sin x)e^y - \cos x \underbrace{\int y e^y \, dy}_{ye^y - e^y = (y-1)e^y} + h(x)$$

$$= -(\cos x - x\sin x - \cos x + y\cos x)e^y + h(x)$$

$$= (x\sin x - y\cos x)e^y + h(x)$$

$$\Rightarrow v_x = (\sin x + x\cos x + y\sin x)e^y + h'(x)$$

$$v_x = -u_y = +(\sin x + x\cos x + y\sin x)e^y$$

$$\Rightarrow h'(x) = 0 \Rightarrow h(x) = h_0 \text{ a constant}$$

$$\Rightarrow v = (x\sin x - y\cos x)e^y + h_0$$

$$v(0,0) = \operatorname{Im} f(0) = 1 \Rightarrow h_0 = 1$$

$$\Rightarrow v = (x\sin x - y\cos x)e^y + 1$$

$$f(x+iy) = -(x\cos x + y\sin x)e^y + i\left[(x\sin x - y\cos x)e^y + 1\right]$$

$$\Rightarrow f(x) = -x\cos x + i(x\sin x + 1) = -x(\cos x - i\sin x) + i$$

$$= -xe^{-x} + i \Rightarrow \boxed{f(z) = -ze^{-z} + i}$$

(7)

Problem 2 / Group 2

2.a $v(x,y) = (y\cos x - x\sin x)\,g(y)$, $\nabla^2 v = 0$

$v_x = [-y\sin x - \sin x - x\cos x]\,g$

$v_{xx} = [-y\cos x - \cos x - \cos x + x\sin x]\,g$

$v_y = (\cos x)\,g + (y\cos x - x\sin x)\,g'$

$v_{yy} = (\cos x)g' + (\cos x)g' + (y\cos x - x\sin x)\,g''$

$\Rightarrow \nabla^2 v = (-y\cos x - 2\cos x + x\sin x)\,g +$

$(2\cos x)\,g' + (y\cos x - x\sin x)\,g''$

$\nabla^2 v = 0 \;\rightarrow\; \cos x\,[-(y+2)g + 2g' + y g''] +$

$x\sin x\,(g - g'') = 0$

$\cos x$ & $x\sin x$ are linearly independent $\Rightarrow$

$y g'' + 2g' - (y+2)g = 0$ (1)

& $g'' - g = 0$ (2)

2.b (1) & (2) $\Rightarrow$ $yg + 2g' - (y+2)g = 0$

$\Rightarrow 2g' - 2g = 0$

$\Rightarrow g' - g = 0$

$\Rightarrow g(y) = ce^{y}$ $c \in \mathbb{R}$

$g(0) = 2 \Rightarrow c = 2 \Rightarrow \boxed{g(y) = 2e^{y}}$

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2.c :

$v = 2(y\cos x - x\sin x)e^{y}$

$v_x = 2(-y\sin x - \sin x - x\cos x)e^{y}$

$v_x = -u_y \Rightarrow u = -\int v_x\, dy + h(x)$

$\Rightarrow u = 2\int (y\sin x + \sin x + x\cos x)e^{y}\, dy + h(x)$

$= 2[\sin x \underbrace{\int y e^{y} dy}_{ye^{y} - e^{y} = (y-1)e^{y}} + (\sin x + x\cos x)e^{y}] + h(x)$

$= 2(y\sin x - \sin x + \sin x + x\cos x)e^{y} + h(x)$

$= 2(y\sin x + x\cos x)e^{y} + h(x)$

$\Rightarrow u_x = 2(y\cos x + \cos x - x\sin x)e^{y} + h'(x)$

$\quad\; \parallel$

$v_y = 2(\cos x + y\cos x - x\sin x)e^{y} \hookrightarrow h'(x) = 0$

$\Downarrow$

$h(x) = h_0 \in \mathbb{R}$

$\Rightarrow u = 2(y\sin x + x\cos x)e^{y} + h_0$

$u(0,0) = Re(f(0)) = -1 \hookrightarrow h_0 = -1$

$\Rightarrow u = 2(y\sin x + x\cos x)e^{y} - 1$

$\Rightarrow f(x+iy) = 2(y\sin x + x\cos x)e^{y} - 1 + i(y\cos x - x\sin x)e^{y}]$

$\Downarrow$

$f(x) = [2x\cos x - 1 + i(-x\sin x)] = 2x(\cos x - i\sin x) - 1$

$= 2xe^{-x} - 1 \Rightarrow \boxed{f(z) = 2ze^{-z} - 1}$

Problem 2

Group 3

2.a $\quad v(x,y) = (y\cos x - x\sin x)\,q(y) + y \quad , \quad \nabla^2 v = 0$

$v_x = [-y\sin x - \sin x - x\cos x\,]\,q(y)$

$v_{xx} = [-y\cos x - \cos x - \cos x + x\sin x\,]\,q(y)$

$v_y = (\cos x)\,q + (y\cos x - x\sin x)\,q' + 1$

$v_{yy} = \cos x\, q' + (\cos x)\,q' + (y\cos x - x\sin x)\,q''$

$$\nabla^2 v = (y\cos x - x\sin x)\,q'' + 2(\cos x)\,q' + \\ -[\,(y+2)\cos x - x\sin x\,]\,q = 0$$

$$\nabla^2 v = 0 \implies \cos x\,[\,y q'' + 2q' - (y+2)q\,] + \\ x\sin x\,[-q'' + q\,] = 0$$

$x\sin x$ & $\cos x$ are linearly independent $\implies$

$$y q'' + 2q' - (y+2)q = 0 \qquad (1)$$

& $\quad q'' - q = 0 \qquad (2)$

2.b $\quad (1)$ & $(2) \implies yq + 2q' - (y+2)q = 0$

$\implies 2q' - 2q = 0 \implies q' = q$

$\implies q(y) = ce^{y}$ for some $c \in \mathbb{R}$

$q(0) = -1 \implies c = -1 \implies \boxed{q(y) = -e^{y}}$

2.c $\quad v = -(y\cos x - x\sin x)e^y + y$

$\Rightarrow v_x = -(-y\sin x - \sin x - x\cos x)e^y$

$= (y\sin x + \sin x + x\cos x)e^y$

$u_y = -v_x \Rightarrow u = \int(-v_x)dy + h(x)$

$\Rightarrow u = -\int(y\sin x + \sin x + x\cos x)e^y\,dy + h(x)$

$= -\sin x \underbrace{\int ye^y dx}_{ye^y - e^y} - (\sin x + x\cos x)\underbrace{\int e^y dy}_{e^y} + h(x)$

$= (-y\sin x + \sin x - \sin x - x\cos x)e^y + h(x)$

$= (-y\sin x - x\cos x)e^y + h(x)$

$\Rightarrow u_x = (-y\cos x - \cos x + x\sin x)e^y + h'(x)$

$\quad \|$

$v_y = -(\cos x + y\cos x - x\sin x)e^y + 1$

$\Rightarrow h'(x) = 1 \Rightarrow h(x) = x + \tilde{c}$ for some $\tilde{c} \in \mathbb{R}$

$\Rightarrow u = -(y\sin x + x\cos x)e^y + x + \tilde{c}$

$u(0,0) = \operatorname{Re}(f(0)) = 2 \;\hookrightarrow\; \tilde{c} = 2$

$\Rightarrow f(x+iy) = -(y\sin x + x\cos x)e^y + x + 2 +$
$\quad + i[-(y\cos x - x\sin x)e^y + y]$

$\Rightarrow f(x) = -x\cos x + x + 2 + i(x\sin x)$

$= -x(\cos x - i\sin x) + 2 \quad = -xe^{-x} + 2$

$\Rightarrow \boxed{f(z) = -ze^{-z} + 2}$

Math 303: Final Exam (Part 3)

May 31, 2020

- This exam consists of five separate 30 minutes-long parts.

- Solve the problem(s) assigned to you. Otherwise you will not be getting any credit.

- Write your name and student ID number in the solution page to each problem.

- Take a photo of the solution page(s) for each problem and upload it to Blackboard as a single electronic file before the due time indicated in the assignment. In addition, send the same file via email to **amostafazadeh@ku.edu.tr** before or within the last three minutes after the end of the time given for each problem. Late submissions of the solution to the problems will not be taken into consideration.

Problem 3 for students with ID numbers: 40600, 49960, 54212, 60333, 64365, 64821

Use contour integration to evaluate $\displaystyle\int_0^\infty \frac{x^{2/3}}{x^2+3}\,dx.$ (25 pts)

Warning: You are expected to give all the details of each step of the necessary calculations. Giving the final numerical value of the integral without the details of the calculations will not earn you any credit.

Problem 3 for students with ID numbers: 50127, 64558, 64869, 54238, 60581, 69720

Use contour integration to evaluate $\displaystyle\int_0^\infty \frac{x^{1/3}}{x^2+5}\,dx.$ (25 pts)

Warning: You are expected to give all the details of each step of the necessary calculations. Giving the final numerical value of the integral without the details of the calculations will not earn you any credit.

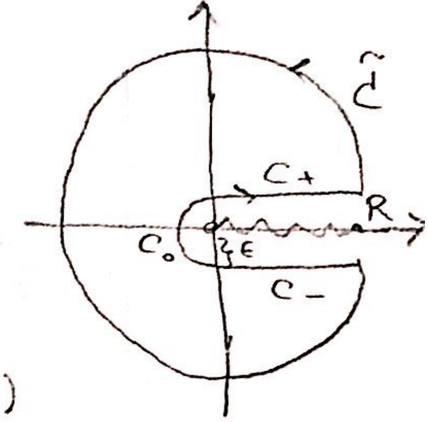
Problem 3 for students with ID numbers: 49642, 64842, 69287, 40557, 53932

Use contour integration to evaluate $\displaystyle\int_0^\infty \frac{x^{1/3}}{x^2+7}\,dx.$ (25 pts)

Warning: You are expected to give all the details of each step of the necessary calculations. Giving the final numerical value of the integral without the details of the calculations will not earn you any credit.

Problem 3 (11)

Group 1



$$I = \int_0^\infty \frac{x^{2/3}}{x^2+3}\,dx \;, \quad f(z) := \frac{z^{2/3}}{z^2+3}$$

$$C = C_+ \sqcup \tilde{C} \sqcup C_- \cup C_0$$

On C_+: $z = x + i\epsilon = \sqrt{x^2+\epsilon^2}\, e^{i\tan^{-1}(\frac{\epsilon}{x})}$

$$\lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_{C_+} f(z)\,dz = I$$

On $\tilde{C}$: $z = Re^{i\theta}$

$$\lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_{\tilde{C}} f(z)\,dz = \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_{\tan^{-1}\frac{\epsilon}{R}}^{2\pi-\tan^{-1}\frac{\epsilon}{R}} \frac{R^{2/3} e^{\frac{2i\theta}{3}}}{R^2 e^{2i\theta}+3}\, iRe^{i\theta}\,d\theta = 0$$

On C_-: $z = x - i\epsilon = \sqrt{x^2+\epsilon^2}\, e^{i(2\pi - \tan^{-1}\frac{\epsilon}{x})}$

$$\Rightarrow \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_{C_-} f(z)\,dz = \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_R^0 \frac{(x^2+\epsilon^2)^{\frac{1}{3}}\, e^{\frac{4\pi i}{3}}\, e^{-\frac{2i}{3}\tan^{-1}\frac{\epsilon}{x}}}{(x-i\epsilon)^2+3}\,dx$$

$$= -e^{\frac{4\pi i}{3}} I = e^{\frac{\pi i}{3}} I$$

On C_0: $z = \epsilon e^{i\theta}$

$$\lim_{\epsilon\to 0} \int_{C_0} f(z)\,dz = \lim_{\epsilon\to 0} \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \frac{\epsilon^{2/3} e^{\frac{2i\theta}{3}}}{\epsilon^2 e^{2i\theta}+3}\, i\epsilon e^{i\theta}\,d\theta = 0$$

$$\Rightarrow \oint_C f(z)\,dz = (1+e^{\frac{\pi i}{3}})\, I \qquad (1)$$

By Residue thm: $\oint_C f(z)\,dz = 2\pi i\,[\,Res(i\sqrt{3}) + Res(-i\sqrt{3})\,]$

$z^2+3=0 \Rightarrow z = \pm i\sqrt{3}$

$$\lim_{z \to i\sqrt{3}} (z - i\sqrt{3}) f(z) = \lim_{z \to i\sqrt{3}} (z - i\sqrt{3}) \frac{z^{2/3}}{(z-i\sqrt{3})(z+i\sqrt{3})}$$

$$= \frac{(i\sqrt{3})^{2/3}}{2i\sqrt{3}} = \frac{(e^{\frac{i\pi}{2}}\sqrt{3})^{2/3}}{2i\sqrt{3}} = \frac{3^{\frac{1}{3}} e^{\frac{i\pi}{3}}}{2i\, 3^{1/2}}$$

$$= \frac{e^{i\pi/3}}{2i\, 3^{1/2 - 1/3}} = \frac{e^{i\pi/3}}{2i\, 3^{1/6}} = Res(i\sqrt{3})$$

$$\lim_{z \to -i\sqrt{3}} (z + i\sqrt{3}) f(z) = \frac{(-i\sqrt{3})^{2/3}}{-2i\sqrt{3}} = \frac{(e^{\frac{3\pi i}{2}} 3^{\frac{1}{2}})^{2/3}}{-2i\, 3^{1/2}}$$

$$= \frac{e^{i\pi}}{-2i\, 3^{\frac{1}{2} - \frac{1}{3}}} = \frac{1}{2i\, 3^{1/6}} = Res(-i\sqrt{3})$$

$$\Rightarrow \oint_C f(z)\, dz = 2\pi i \left(\frac{e^{i\pi/3}}{2i\, 3^{1/6}} + \frac{1}{2i\, 3^{1/6}} \right) = \frac{\pi}{3^{1/6}} \left(e^{i\pi/3} + 1 \right)$$

(2)

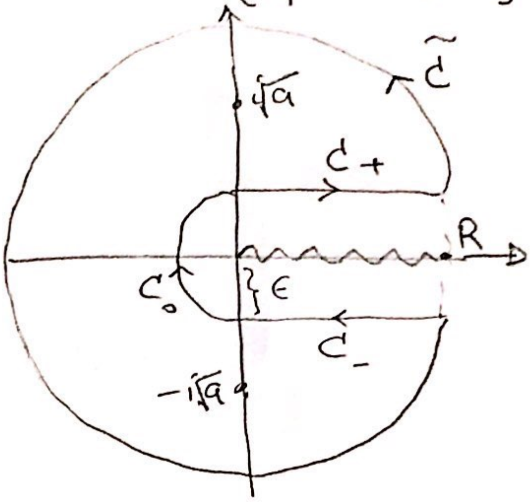
(1) & (2) :

$$(1 + e^{\frac{i\pi}{3}})\, I = \frac{\pi}{3^{1/6}} (1 + e^{i\pi/3})$$

$$\Rightarrow \boxed{I = \frac{\pi}{3^{1/6}}}$$

Problem 3

Group 2 ve 3 : $\quad I = \int_0^\infty \frac{x^{\frac{1}{3}}}{x^2+a}\,dx \qquad a = \begin{cases} 5 & \text{Group 2} \\ 7 & \text{" 3} \end{cases}$



$f(z) := \frac{z^{1/3}}{z^2+a}$

$C = C_+ \sqcup \tilde{C} \sqcup C_- \sqcup C_0$

On C_+: $\quad z = x + i\epsilon = \sqrt{x^2+\epsilon^2}\, e^{i\tan^{-1}(\frac{\epsilon}{x})}$

$\lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_{C_+} f(z)\,dz = I$

on $\tilde{C}$: $\quad z = Re^{i\theta}$

$\lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_{\tilde{C}} f(z)\,dz = \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_{\tan^{-1}(\frac{\epsilon}{R})}^{2\pi - \tan^{-1}(\frac{\epsilon}{R})} \frac{R^{1/3} e^{\frac{i\theta}{3}}}{R^2 e^{2i\theta}+a}\, iRe^{i\theta}\,d\theta = 0$

On C_- : $\quad z = x - i\epsilon = \sqrt{x^2+\epsilon^2}\; e^{i[2\pi - \tan^{-1}(\frac{\epsilon}{x})]}$

$\lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_{C_-} f(z)\,dz = \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_R^0 \frac{(x^2+\epsilon^2)^{\frac{1}{6}}\, e^{\frac{2\pi i}{3}}\, e^{-\frac{i}{3}\tan^{-1}(\frac{\epsilon}{x})}}{(x-i\epsilon)^2+a}\,dx$

$= -e^{\frac{2\pi i}{3}}\, I = e^{-\frac{i\pi}{3}}\, I$

On C_0 : $\quad z = \epsilon e^{i\theta}$

$\lim_{\epsilon\to 0} \int_{C_0} f(z)\,dz = \lim_{\epsilon\to 0} \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \frac{\epsilon^{1/3} e^{\frac{i\theta}{3}}}{\epsilon^2 e^{2i\theta}+a}\, i\epsilon e^{i\theta}\,d\theta = 0$

$\Rightarrow \oint_C f(z)\,dz = (1 + e^{-\frac{i\pi}{3}})\, I \qquad (1)$

By residue thm: $\oint_C f(z)\,dz = 2\pi i\,[\operatorname{Res}(i\sqrt{a}) + \operatorname{Res}(-i\sqrt{a})$

$z^2 + a = 0 \;\Rightarrow\; z = \pm i\sqrt{a} = \begin{cases} \sqrt{a}\, e^{i\pi/2} \\ \sqrt{a}\, e^{3i\pi/2} \end{cases}$

$$\lim_{z\to\pm i\sqrt{a}} (z\mp i\sqrt{a})f(z) = \lim_{z\to\pm i\sqrt{a}} (z\mp i\sqrt{a})\frac{z^{1/3}}{(z-i\sqrt{a})(z+i\sqrt{a})}$$

$$= \frac{(\pm i\sqrt{a})^{1/3}}{\pm 2i\sqrt{a}} = \pm\frac{(\pm i)^{1/3}a^{\frac{1}{6}}}{2ia^{\frac{1}{2}}} = \pm\frac{(\pm i)^{1/3}}{2ia^{1/3}}$$

$\Rightarrow$ $\pm i\sqrt{a}$ are simple poles &

$$\text{Res}(i\sqrt{a}) = \frac{i^{1/3}}{2ia^{1/3}} = \frac{e^{\frac{i\pi}{6}}}{2ia^{1/3}} = \frac{e^{\frac{i\pi}{6}}}{2e^{i\pi/2}a^{1/3}} = \frac{e^{-i\pi/3}}{2a^{1/3}}$$

$$\text{Res}(-i\sqrt{a}) = -\frac{(-i)^{1/3}}{2ia^{1/3}} = -\frac{\left(e^{\frac{3i\pi}{2}}\right)^{\frac{1}{3}}}{2ia^{1/3}} = -\frac{e^{\frac{i\pi}{2}}}{2ia^{1/3}} = -\frac{1}{2a^{1/3}}$$

$$\Rightarrow \oint_C f(z)\,dz = 2\pi i\left(\frac{e^{-i\pi/3}}{2a^{1/3}} - \frac{1}{2a^{1/3}}\right) = \frac{\pi i}{a^{1/3}}\left(e^{-i\pi/3}-1\right)$$

(1) & (2) : $\quad (1+e^{-\frac{i\pi}{3}})I = \frac{\pi i}{a^{1/3}}(e^{-i\pi/3}-1)$

$$I = \frac{\pi i}{a^{1/3}}\left(\frac{e^{-i\pi/3}-1}{e^{-i\pi/3}+1}\right) = \frac{\pi i}{a^{1/3}}\left[\frac{e^{-\frac{i\pi}{6}}(e^{-\frac{i\pi}{6}}-e^{i\pi/6})}{e^{-\frac{i\pi}{6}}(e^{-\frac{i\pi}{6}}+e^{i\pi/6})}\right]$$

$$= \frac{\pi i}{a^{1/3}}\left[\frac{-2i\sin(\frac{\pi}{6})}{2\cos(\frac{\pi}{6})}\right] = \frac{\pi}{a^{1/3}}\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$= \frac{\pi}{\sqrt{3}\,a^{1/3}}.$$

Math 303: Final Exam (Part 4)

May 31, 2020

- This exam consists of five separate 30 minutes-long parts.

- Solve the problem(s) assigned to you. Otherwise you will not be getting any credit.

- Write your name and student ID number in the solution page to each problem.

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Problem 4 for students with ID numbers: 40600, 49960, 54212, 60333, 64365, 64821

Let for all $x \in \mathbb{R}$, $\theta(x) := \begin{cases} 0 \text{ for } x < 0 \\ 1 \text{ for } x \geq 0 \end{cases}$, $f(x) := e^{x}\theta(1-4x^2)$, and $g(x) := e^{-2x}f'(x)$.

4.a (7 pts) Express $g(x)$ in terms of the step function θ and the Dirac delta function δ. Simplify your response as much as possible.

4.b (10 pts) Let $\tilde{g}(k)$ be the Fourier transform of $g(x)$. Find an explicit expression for $\tilde{g}(k)$ and simplify it as much as possible.

4.c (3 pts) Calculate $\tilde{g}(0)$.

Problem 4 for students with ID numbers: 50127, 64558, 64869, 54238, 60581, 69720

Let for all $x \in \mathbb{R}$, $\theta(x) := \begin{cases} 0 \text{ for } x < 0 \\ 1 \text{ for } x \geq 0 \end{cases}$, $f(x) := e^{-x}\theta(2-x^2/2)$, and $g(x) := e^{x/2}f'(x)$.

4.a (7 pts) Express $g(x)$ in terms of the step function θ and the Dirac delta function δ. Simplify your response as much as possible.

4.b (10 pts) Let $\tilde{g}(k)$ be the Fourier transform of $g(x)$. Find an explicit expression for $\tilde{g}(k)$ and simplify it as much as possible.

4.c (3 pts) Calculate $\tilde{g}(0)$.

Problem 4 for students with ID numbers: 49642, 64842, 69287, 40557, 53932

Let for all $x \in \mathbb{R}$, $\theta(x) := \begin{cases} 0 \text{ for } x < 0 \\ 1 \text{ for } x \geq 0 \end{cases}$, $f(x) := e^{x/2}\theta(1-x^2/4)$, and $g(x) := e^{-x/2}f'(x)$.

4.a (7 pts) Express $g(x)$ in terms of the step function θ and the Dirac delta function δ. Simplify your response as much as possible.

4.b (10 pts) Let $\tilde{g}(k)$ be the Fourier transform of $g(x)$. Find an explicit expression for $\tilde{g}(k)$ and simplify it as much as possible.

4.c (3 pts) Calculate $\tilde{g}(0)$.

Problem 4

Group 1

4.a $f = e^{x}\theta(1-4x^2)$

$$f' = e^{x}\theta(1-4x^2) + e^{x}(-8x)\delta(1-4x^2)$$

$$= e^{x}[\theta(1-4x^2) - 8x\,\delta(1-4x^2)]$$

$h(x) = 4x^2-1 \Rightarrow \delta(1-4x^2) = \delta(4x^2-1) = \delta(h(x))$

$h'(x) = 8x$

$h(x) = 0 \Rightarrow x = \pm\frac{1}{2}$ $\hookrightarrow$ $\delta(1-x^2) = \dfrac{\delta(x-\frac{1}{2})}{|h'(\frac{1}{2})|} + \dfrac{\delta(x+\frac{1}{2})}{|h'(-\frac{1}{2})|}$

$h'(\pm\frac{1}{2}) = \pm 4$

$$\delta(1-4x^2) = \frac{\delta(x-\frac{1}{2})}{4} + \frac{\delta(x+\frac{1}{2})}{4} = \frac{1}{4}[\delta(x-\tfrac{1}{2}) + \delta(x+\tfrac{1}{2})]$$

$$\Rightarrow f'(x) = e^{x}\{\theta(1-4x^2) - 2x[\delta(x-\tfrac{1}{2}) + \delta(x+\tfrac{1}{2})]\}$$

$$= e^{x}[\theta(1-4x^2) - \delta(x-\tfrac{1}{2}) + \delta(x+\tfrac{1}{2})]$$

$$\Rightarrow g = e^{-2x}f'(x) = e^{-x}[\theta(1-4x^2) - \delta(x-\tfrac{1}{2}) + \delta(x+\tfrac{1}{2})]$$

$$= e^{-x}\theta(1-4x^2) - e^{-1/2}\delta(x-\tfrac{1}{2}) + e^{1/2}\delta(x+\tfrac{1}{2})$$

4.b $\tilde{g}(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iux} g(x)\,dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{(-iu-1)x}[\theta(1-4x^2) - \delta(x-\tfrac{1}{2}) + \delta(x+\tfrac{1}{2})]\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-(iu+1)x}\,dx - e^{-\frac{(iu+1)}{2}} + e^{+\frac{(iu+1)}{2}}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left.\frac{e^{-(iu+1)x}}{-(iu+1)}\right|_{-\frac{1}{2}}^{\frac{1}{2}} - e^{-\frac{iu+1}{2}} + e^{\frac{iu+1}{2}}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[-\left(1+\frac{1}{iu+1}\right)e^{-\frac{iu+1}{2}} + \left(1+\frac{1}{iu+1}\right)e^{\frac{iu+1}{2}}\right]$$

$$\Rightarrow \tilde{g}(k) = \frac{1}{\sqrt{2\pi}} \left(1 + \frac{1}{ik+1}\right)\left(e^{\frac{ik+1}{2}} - e^{-\frac{ik+1}{2}}\right)$$

$$= \sqrt{\frac{2}{\pi}} \left(1 + \frac{1}{ik+1}\right) \sinh\left[\left(\frac{ik+1}{2}\right)\right]$$

4.c $\quad \tilde{g}(0) = \sqrt{\frac{2}{\pi}}\, 2 \sinh\left(\frac{1}{2}\right).$

Problem 4

Group 2

4.a : $\quad f = e^{-x}\,\theta\left(2 - \frac{x^2}{2}\right)$

$$\Rightarrow f' = -e^{-x}\,\theta\left(2-\frac{x^2}{2}\right) - x e^{-x}\,\delta\left(2-\frac{x^2}{2}\right)$$

$$= -e^{-x}\left[\theta\left(2-\frac{x^2}{2}\right) + x\,\delta\left(2-\frac{x^2}{2}\right)\right]$$

$h(x) := 2 - \frac{x^2}{2} \Rightarrow h(x) = 0 \Rightarrow x = \pm 2$

$h'(x) = -x \Rightarrow h'(\pm 2) = \mp 2$

$$\delta\left(2-\frac{x^2}{2}\right) = \frac{\delta(x-2)}{|h'(2)|} + \frac{\delta(x+2)}{|h'(-2)|} = \frac{1}{2}\left[\delta(x-2) + \delta(x+2)\right]$$

$$\Rightarrow g(x) = e^{\frac{x}{2}} f'(x) = -e^{-\frac{x}{2}}\left\{\theta\left(2-\frac{x^2}{2}\right) + \frac{x}{2}\left[\delta(x-2) + \delta(x+2)\right]\right\}$$

$$= -e^{-\frac{x}{2}}\left[\theta\left(2-\frac{x^2}{2}\right) + \delta(x-2) - \delta(x+2)\right]$$

$$= -e^{-\frac{x}{2}}\,\theta\left(2-\frac{x^2}{2}\right) - e^{-1}\delta(x-2) + e\,\delta(x+2)$$

4.b $\quad \tilde{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\left[-e^{-\frac{x}{2}}\theta\left(2-\frac{x^2}{2}\right) - \frac{1}{e}\delta(x-2) + e\,\delta(x+2)\right] dx$

$$\Rightarrow \tilde{g}(k) = \frac{1}{\sqrt{2\pi}}\left[\int_{-2}^{2} -e^{-(ik+\frac{1}{2})x}\, dx - \frac{1}{e} e^{-2ik} + e\, e^{2ik}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left.\frac{e^{-(ik+\frac{1}{2})x}}{ik+\frac{1}{2}}\right|_{-2}^{2} + e^{2ik+1} - e^{-(2ik+1)}\right.$$

$$\Rightarrow \tilde{g}(k) = \frac{1}{\sqrt{2\pi}} \left\{ \left(\frac{1}{ik+\frac{1}{2}} - 1\right) e^{-(2ik+1)} - \left(\frac{1}{ik+\frac{1}{2}} - 1\right) e^{2ik+1} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left(1 - \frac{1}{ik+\frac{1}{2}}\right)\left(e^{2ik+1} - e^{-(2ik+1)}\right)$$

$$= \sqrt{\frac{2}{\pi}} \left(1 - \frac{1}{ik+\frac{1}{2}}\right) \sinh(2ik+1)$$

4.c $\quad \tilde{g}(0) = \sqrt{\frac{2}{\pi}}\,(1-2)\sinh(1) = -\sqrt{\frac{2}{\pi}}\sinh(1).$

Problem 4

Group 3

4.a: $\quad f = e^{\frac{x}{2}}\,\theta\left(1 - \frac{x^2}{4}\right)$

$$\Rightarrow f' = \frac{1}{2} e^{\frac{x}{2}} \theta\left(1-\frac{x^2}{4}\right) + e^{\frac{x}{2}}\left(-\frac{x}{2}\right)\delta\left(1-\frac{x^2}{4}\right)$$

$$= \frac{e^{\frac{x}{2}}}{2}\left[\theta\left(1-\frac{x^2}{4}\right) - x\,\delta\left(1-\frac{x^2}{4}\right)\right]$$

$h(x) := 1 - \frac{x^2}{4} \Rightarrow h(0) \Rightarrow x = \pm 2$

$h'(x) = -\frac{x}{2} \Rightarrow h(\pm 2) = \mp 1$

$$\delta\left(1-\frac{x^2}{4}\right) = \frac{\delta(x-2)}{|h'(2)|} + \frac{\delta(x+2)}{|h'(-2)|} = \delta(x-2) + \delta(x+2)$$

$$\Rightarrow g(x) = e^{-\frac{x}{2}}\left\{\frac{e^{\frac{x}{2}}}{2}\left[\theta\left(1-\frac{x^2}{4}\right) - x\left[\delta(x-2)+\delta(x+2)\right]\right]\right\}$$

$$= \frac{1}{2}\theta\left(1-\frac{x^2}{4}\right) - \frac{x}{2}\left[\delta(x-2) + \delta(x+2)\right]$$

$$= \frac{1}{2}\theta\left(1-\frac{x^2}{4}\right) - \delta(x-2) + \delta(x+2)$$

4.b $$\tilde{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \left\{ \frac{1}{2}\Theta(1-\frac{x^2}{4}) - \delta(x-2) + \delta(x+2) \right\} dx$$

$$\Rightarrow \tilde{g}(k) = \frac{1}{\sqrt{2\pi}} \left\{ \frac{1}{2} \int_{-2}^{2} e^{-ikx}\, dx - e^{-2ik} + e^{2ik} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{2} \left(\frac{e^{-ikx}}{-ik} \right|_{-2}^{2} + e^{2ik} - e^{-2ik} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-2ik}}{-2ik} + \frac{e^{2ik}}{2ik} + e^{2ik} - e^{-2ik} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[(1+\frac{1}{2ik}) e^{2ik} - (1+\frac{1}{2ik}) e^{-2ik} \right]$$

$$= \frac{1}{\sqrt{2\pi}} (1+\frac{1}{2ik})(e^{2ik} - e^{-2ik})$$

$$= \sqrt{\frac{2}{\pi}}\, i(1+\frac{1}{2ik}) \sin(2k)$$

$$= \sqrt{\frac{2}{\pi}} (\frac{1}{2k}+i) \sin(2k)$$

4.c $\tilde{g}(0)$ is not defind but.

$$\lim_{k\to 0} \tilde{g}(k) = \sqrt{\frac{2}{\pi}}$$

Math 303: Final Exam (Part 5)

May 31, 2020

- This exam consists of five separate 30 minutes-long parts.

- Solve the problem(s) assigned to you. Otherwise you will not be getting any credit.

- Write your name and student ID number in the solution page to each problem.

- Take a photo of the solution page(s) for each problem and upload it to Blackboard as a single electronic file before the due time indicated in the assignment. In addition, send the same file via email to **amostafazadeh@ku.edu.tr** before or within the last three minutes after the end of the time given for each problem. Late submissions of the solution to the problems will not be taken into consideration.

Problem 5 for students with ID numbers: 40600, 49960, 54212, 60333, 64365, 64821

Use the method of Fourier transform to find a particular solution of the following forth order differential equation:

$$y^{(4)}(x) - y'(x) - 4y(x) = \sin(2x). \qquad \text{(20 pts)}$$

You are expected to simplify this solution as much as possible and express it as a manifestly real-valued function of x.

Problem 5 for students with ID numbers: 50127, 64558, 64869, 54238, 60581, 69720

Use the method of Fourier transform to find a particular solution of the following forth order differential equation:

$$y^{(4)}(x) + y'(x) - 4y(x) = \cos(2x). \qquad \text{(20 pts)}$$

You are expected to simplify this solution as much as possible and express it as a manifestly real-valued function of x.

Problem 5 for students with ID numbers: 49642, 64842, 69287, 40557, 53932

Use the method of Fourier transform to find a particular solution of the following forth order differential equation:

$$y^{(4)}(x) + 4y'(x) - y(x) = 4\sin(x). \qquad \text{(20 pts)}$$

You are expected to simplify this solution as much as possible and express it as a manifestly real-valued function of x.

Problem 5

Group 1 $\qquad Y^{(4)}(x) - Y'(x) - 4Y(x) = \sin(2x)$

$$\tilde{Y}(k) := \mathcal{F}\{Y(x)\} \Rightarrow (ik)^4\tilde{Y} - (ik)\tilde{Y} - 4\tilde{Y} = \mathcal{F}\{\sin(2x)\}$$

$$\Rightarrow (k^4 - ik + 4)\tilde{Y}(k) = \frac{1}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty} e^{-ikx}\left[\frac{e^{2ix} - e^{-2ix}}{2i}\right]dx}_{\frac{1}{2i}\underbrace{\int_{-\infty}^{\infty}\left[e^{-i(k-2)x} - e^{-i(k+2)x}\right]dx}_{2\pi[\delta(k-2) - \delta(k+2)]}}$$

$$\Rightarrow \tilde{Y}(k) = -\sqrt{\frac{\pi}{2}}\, i\left[\frac{\delta(k-2) - \delta(k+2)}{k^4 - ik + 4}\right]$$

$$\Rightarrow Y(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\left\{-\sqrt{\frac{\pi}{2}}\, i\left[\frac{\delta(k-2) - \delta(k+2)}{k^4 - ik + 4}\right]\right\}dk$$

$$= -\frac{i}{2}\left[\frac{e^{2ix}}{16 - 2i + 4} - \frac{e^{-2ix}}{16 + 2i + 4}\right]$$

$$= -\frac{i}{4}\left(\frac{e^{2ix}}{10 - i} - \frac{e^{-2ix}}{10 + i}\right)$$

$$= -\frac{i}{4}\left[\frac{(10+i)e^{2ix} - (10-i)e^{-2ix}}{100 + 1}\right]$$

$$= -\frac{i}{4}\left[\frac{10(e^{2ix} - e^{-2ix}) + i(e^{2ix} + e^{-2ix})}{101}\right]$$

$$= \frac{1}{202}\left[10\sin(2x) + \cos(2x)\right]$$

Problem 5

Group 2

$$Y^{(4)}(x) - Y'(x) - 4Y(x) = \cos(2x)$$

$$\tilde{Y}(k) := \mathcal{F}\{y(x)\} \rightarrow (ik)^4 \tilde{Y}(k) - ik\,\tilde{Y}(k) - 4\,\tilde{Y}(k) = \mathcal{F}\{\cos(2x)\}$$

$$\Rightarrow (k^4 - ik - 4)\,\tilde{Y}(k) = \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-ikx} \, \frac{1}{2}\left(e^{2ix} + e^{-2ix}\right) dx}_{\frac{1}{2} \underbrace{\int_{-\infty}^{\infty} \left[e^{-i(k-2)x} + e^{-i(k+2)}\right] dx}_{2\pi\left[\delta(k-2) + \delta(k+2)\right]}}$$

$$\Rightarrow \tilde{Y}(k) = \sqrt{\frac{\pi}{2}} \left[\frac{\delta(k-2) + \delta(k+2)}{k^4 - ik - 4} \right]$$

$$\Rightarrow Y(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \sqrt{\frac{\pi}{2}} \left[\frac{\delta(k-2) + \delta(k+2)}{k^4 - ik - 4} \right] dk$$

$$= \frac{1}{2} \left[\frac{e^{2ix}}{16 - 2i - 4} + \frac{e^{-2ix}}{16 + 2i - 4} \right]$$

$$= \frac{1}{4} \left(\frac{e^{2ix}}{6 - i} + \frac{e^{-2ix}}{6 + i} \right)$$

$$= \frac{1}{4} \left[\frac{(6+i)\,e^{2ix} + (6-i)\,e^{-2ix}}{36 + 1} \right]$$

$$= \frac{1}{4 \times 37} \left[6\left(e^{2ix} + e^{-2ix}\right) + i\left(e^{2ix} - e^{-2ix}\right) \right]$$

$$= \frac{1}{74} \left[6\cos(2x) - \sin(2x) \right]$$

Problem 5

Group 3

$$y^{(4)}(x) + 4y'(x) - y(x) = 4\sin x$$

$$\tilde{Y}(k) := \mathcal{F}\{y(x)\} \hookrightarrow (ik)^4 \tilde{Y} + 4ik\tilde{Y} - \tilde{Y} = \mathcal{F}\{\sin x\}$$

$$\Rightarrow (k^4 + 4ik - 1)\tilde{Y}(k) = \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-ikx} \left(\frac{e^{ix} - e^{-ix}}{2i} \right) dx}_{\frac{1}{2i} \underbrace{\int_{-\infty}^{\infty} [e^{-i(k-1)x} - e^{-i(k+1)x}] dx}_{2\pi[\delta(k-1) - \delta(k+1)]}}$$

$$\Rightarrow \tilde{Y}(k) = -\sqrt{\frac{\pi}{2}}\, i \left[\frac{\delta(k-1) - \delta(k+1)}{k^4 + 4ik - 1} \right]$$

$$\Rightarrow y(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(-\sqrt{\frac{\pi}{2}}\right) i \left[\frac{\delta(k-1) - \delta(k+1)}{k^4 + 4ik - 1} \right] e^{ikx} \, dk$$

$$= -\frac{i}{2} \left[\frac{e^{ix}}{1 + 4i - 1} - \frac{e^{-ix}}{1 - 4i - 1} \right]$$

$$= -\frac{1}{8} \left(e^{ix} + e^{-ix} \right)$$

$$= -\frac{\cos x}{4}.$$