Math 303: Final Exam

May 28, 2019

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 2.5 hours.
- You must show the details of all your work. Illegible and ambiguous explanations and calculations will lead to deductions from your grade.

Problem 1 The exponential and natural logarithm of a complex number z are defined as follows.

- Let $x := \mathrm{Re}(z)$ and $y := \mathrm{Im}(z)$, then $e^z := e^x(\cos y + i \sin y)$.
- $\ln z$ is defined to be a complex number w satisfying $e^w = z$.

1.a (5 points) Use the above definition of e^z and properties of the exponential and trigonometric functions of a real variable to show that for all $z_1, z_2 \in \mathbb{C}$, $e^{z_1}e^{z_2} = e^{z_1+z_2}$.

1.b (5 points) Use the above definition of $\ln z$ to show that for all $z_1, z_2 \in \mathbb{C}$, $\ln(z_1 z_2) = \ln z_1 + \ln z_2$.

1.c (10 points) Find the imaginary part of all possible values of $(\ln z)^{\ln z}$ for $z := \sqrt{\frac{e}{2}}\,(1+i)$.

Problem 2 (15 points) Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be a solution of the Laplace equation, $\nabla^2\psi = 0$, in $\mathbb{R}^2$. Show that the stationary points of ψ are isolated points of $\mathbb{R}^2$, i.e., given any stationary point of ψ there is a disc centered at this point that contains no other stationary point of ψ.

Problem 3 Let $f : \mathbb{C} \to \mathbb{C}$ be a complex-valued function with domain $\mathbb{C}$, and $u, v : \mathbb{R}^2 \to \mathbb{R}$ be functions defined by $u(x, y) := \mathrm{Re}(f(x + iy))$ and $v(x, y) := \mathrm{Im}(f(x + iy))$.

3.a (5 points) Give the definition of the derivative of f at a point $z \in \mathbb{C}$.

3.b (10 points) Suppose that f is a differentiable function, and u and v have continuous second order partial derivatives at every $(x, y) \in \mathbb{R}^2$. Show that the derivative of f is differentiable.

Problem 4 Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be the functions defined by

$$f(x) := \begin{cases} e^{2x} & \text{for} \quad x < 0, \\ 1 & \text{for} \quad 0 \le x < 1, \\ e^{3(1-x)} & \text{for} \quad x \ge 1, \end{cases} \qquad\qquad g(x) := \frac{d^2}{dx^2} f(x) + \frac{d}{dx} f(x) - 6f(x).$$

4.a (10 points) Find an explicit expression for $g(x)$ for all $x \in \mathbb{R}$.

4.b (10 points) Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{g(x)}{x+1}\, dx$.

Problem 5 (30 points) Evaluate the Fourier transform of $f(x) := (x^2+1)^{-2}$.
Note: To get proper credit, you must use the methods and results covered in this course.

Math 303: Soln's to Final Exam

problem 1.a) $\quad z_1 = x_1 + i y_1 , \quad z_2 = x_2 + i y_2$

$$e^{z_1} e^{z_2} = e^{x_1}(\cos y_1 + i \sin y_1)\, e^{x_2}(\cos y_2 + i \sin y_2)$$

$$= e^{x_1 + x_2}\left[\cos y_1 \cos y_2 - \sin y_1 \sin y_2 + i(\sin y_1 \cos y_2 + \cos y_1 \sin y_2)\right]$$

$$= e^{x_1+x_2}\left[\cos(y_1+y_2) + i \sin(y_1+y_2)\right]$$

$$= e^{x_1 + x_2 + i(y_1+y_2)}$$

$$= e^{z_1+z_2}$$

1.b) let $w_1 = \ln z_1 , \quad w_2 = \ln z_2 \Rightarrow$

$e^{w_1} = z_1$ and $e^{w_2} = z_2$

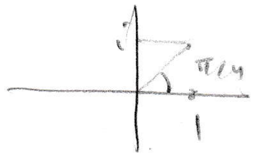
$\Rightarrow \quad z_1 z_2 = e^{w_1} e^{w_2} = e^{w_1 + w_2}$

$\Rightarrow \quad w_1 + w_2 = \ln(z_1 z_2)$

$\Rightarrow \quad \ln z_1 + \ln z_2 = \ln(z_1 + z_2)$

1.c)

$$z = \sqrt{\frac{e}{2}}\,(1+i)$$



$$= \sqrt{e}\; e^{\frac{i\pi}{4}}$$

$$\ln z = \ln\left(\sqrt{e}\; e^{i\pi/4 + 2\pi n i}\right) = \ln\sqrt{e} + i\pi\left(\frac{1}{4} + 2n\right)$$

$$= \frac{1}{2} + i\pi\left(\frac{1}{4} + 2n\right), \quad n \in \mathbb{Z}$$

$$(\ln z)^{\ln z} = e^{\ln z \ln(\ln z)}$$

$$\ln(\ln z) = \ln\left(\frac{1}{2} + i\pi\left(\frac{1}{4} + 2n\right)\right)$$

$$\left|\frac{1}{2} + i\pi\left(\frac{1}{4} + 2n\right)\right| = \sqrt{\frac{1}{4} + \pi^2\left(\frac{1}{4} + 2n\right)^2} =: a_n$$

$$\varphi_n := \tan^{-1}\left(\frac{\pi\left(\frac{1}{4} + 2n\right)}{\frac{1}{2}}\right) = \tan^{-1}\left(\frac{\pi}{2} + 4\pi n\right)$$

$$\Rightarrow \ln(\ln z) = \ln\left[a_n\, e^{i\varphi_n + 2\pi m i}\right]$$

$$= \ln(a_n) + i(\varphi_n + 2\pi m)$$

$$(\ln z)^{\ln z} = e^{\left[\frac{1}{2} + i\pi\left(\frac{1}{4} + 2n\right)\right]\left[\ln(a_n) + i(\varphi_n + 2\pi m)\right]}$$

$$= e^{\frac{1}{2}\ln a_n - \pi\left(\frac{1}{4} + 2n\right)(\varphi_n + 2\pi m)} \times e^{i\left[\pi\left(\frac{1}{4} + 2n\right)\ln a_n + \frac{1}{2}(\varphi_n + 2\pi m)\right]}$$

$$\boxed{\mathrm{Im}\left[(\ln z)^{\ln z}\right] = e^{\frac{1}{2}\ln a_n - \pi\left(\frac{1}{4} + 2n\right)(\varphi_n + 2\pi m)}\; \sin\left[\pi\left(\frac{1}{4} + 2n\right)\ln a_n + \frac{1}{2}(\varphi_n + 2\pi m)\right]}$$

Problem 2 : $\nabla^2 \psi = 0 \Rightarrow \psi_{xx} + \psi_{yy} = 0$

let $u := \psi_x \Rightarrow u_x = \psi_{xx} = -\psi_{yy}$

So let $v = -\psi_y \hookrightarrow \boxed{u_x = v_y}$

& $u_y = \psi_{yx} = -v_x \Rightarrow \boxed{u_y = -v_x}$

$\Downarrow$

$f(x+iy) := u(x,y) + i v(x,y)$

$= \psi_x(x,y) - i\psi_y(x,y)$

$\Rightarrow f: \mathbb{C} \to \mathbb{C}$ is holomorphic function

$\Downarrow$

Its zeros are isolated.

$f(z) = 0 \iff \psi_x(x,y) = 0$ & $\psi_y(x,y) = 0$

$\iff \vec{\nabla}\psi(x,y) = \vec{0}$

$\iff (x,y)$ is a stationary point of ψ

Stationary point of ψ are isolated points of $\mathbb{R}^2$.

Problem 3.a)

$$f'(z) = \lim_{w \to 0} \frac{f(z+w) - f(z)}{w}$$

when $w \in \mathbb{C}$.

3.b) Let $x, y \in \mathbb{R}$, $z = x + iy$

$f(z) = f(x+iy)$ & $f'(x+iy) = \frac{\partial}{\partial x}[u(x,y) + iv(x,y)]$

let $\tilde{u} := u_x$ and $\tilde{v} := v_x$ so that

$f'(x+iy) = \tilde{u}(x,y) + i\tilde{v}(x,y)$

It is sufficient to show that $\tilde{u}$ & $\tilde{v}$ fulfil the Cauchy-Riemann conditions.

Because f is a differentiable function, u and v satisfy the Cauchy-Riemann conditions, i.e.,

$u_x = v_y$ and $u_y = -v_x$

$\Downarrow$

$\tilde{u}_x = u_{xx} = v_{yx} = v_{xy} = \tilde{v}_y$

$\tilde{u}_y = u_{xy} = u_{yx} = -v_{xx} = -\tilde{v}_x$

$\Rightarrow$ $\tilde{u}$ & $\tilde{v}$ fulfil the Cauchy-Riemann Condition

$\Downarrow$

f' is differentiable.

Problem 4.a:

$$f(x) = e^{2x}\,\theta(-x) + \theta(x) - \theta(x-1) + e^{3(1-x)}\,\theta(x-1)$$

$$= e^{2x}\,\theta(-x) + \theta(x) + \left[e^{3(1-x)} - 1\right]\theta(x-1)$$

$$\frac{d}{dx}f(x) = 2e^{2x}\,\theta(-x) - e^{2x}\,\delta(-x) + \delta(x) +$$
$$- 3e^{3(1-x)}\,\theta(x-1) + \left[e^{3(1-x)} - 1\right]\delta(x-1)$$

$$= 2e^{2x}\,\theta(-x) - \delta(x) + \delta(x) - 3e^{3(1-x)}\,\theta(x-1)$$

$$= 2e^{2x}\,\theta(-x) - 3e^{3(1-x)}\,\theta(x-1)$$

$$\frac{d^2}{dx^2}f(x) = 4e^{2x}\,\theta(-x) - 2e^{2x}\,\delta(-x) + 9e^{3(1-x)}\,\theta(x-1)$$
$$- 3e^{3(1-x)}\,\delta(x-1)$$

$$= 4e^{2x}\,\theta(-x) - 2\delta(x) + 9e^{3(1-x)}\,\theta(x-1) - 3\delta(x-1)$$

$$g(x) = 4e^{2x}\,\theta(-x) + 9e^{3(1-x)}\,\theta(x-1) - 2\delta(x) - 3\delta(x-1)$$
$$+ 2e^{2x}\,\theta(-x) - 3e^{3(1-x)}\,\theta(x-1) +$$
$$- 6\left[e^{2x}\,\theta(1-x) + e^{3(1-x)}\,\theta(x-1) + \theta(x) - \theta(x-1)\right]$$

$$= -2\delta(x) - 3\delta(x-1) - 6\left[\theta(x) - \theta(x-1)\right]$$

4.b) $\displaystyle\int_{-\infty}^{\infty} \frac{g(x)}{x+1}\,dx =$

$$\int_{-\infty}^{\infty} \frac{-2\,\delta(x) - 3\,\delta(x-1) - 6\,[\theta(x) - \theta(x-1)]}{x+1}\,dx$$

$$= -\frac{2}{1} - \frac{3}{1+1} - 6\int_0^1 \frac{dx}{x+1}$$

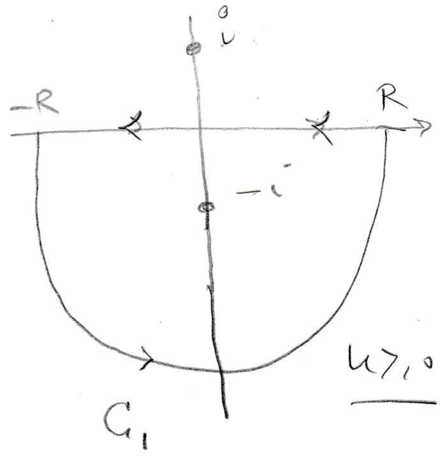
$$= -2 - \frac{3}{2} - 6\,\ln(x+1)\Big|_0^1$$

$$= -2 - \frac{3}{2} - 6\ln 2$$

$$= -\left(\frac{7}{2} + 6\ln 2\right)$$

Problem 5 : $\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{(x^2+1)^2}\, dx$

Let $g(z) := \frac{e^{-ikz}}{(z^2+1)^2}$



$z^2+1=0 \Rightarrow z = \pm i$ double poles

For $k \geq 0$:

$k \geq 0$

$$\underbrace{\lim_{R\to\infty} \oint_{C} g(z)\,dz}_{I} = \lim_{R\to\infty} \Big[\int_{C_1} g(z)\,dz + \underbrace{\int_{R}^{-R} g(x)\,dx}_{-\sqrt{2\pi}\,\tilde{f}(k)} \Big]$$

On C_1, $\operatorname{Im}(z) < 0 \Rightarrow \lim_{R\to\infty} g(z) = 0$

$$\tilde{f}(k) = \frac{-1}{\sqrt{2\pi}} \lim_{R\to\infty} \oint g(z)\,dz = \frac{-2\pi i}{\sqrt{2\pi}} \operatorname{Res}(-i)$$

$$g(z) = \frac{e^{-ikz}}{(z+i)^2(z-i)^2} \qquad \text{let} \quad h(z) = \frac{e^{-ikz}}{(z-i)^2}$$

$$\Rightarrow g(z) = \frac{h(z)}{(z+i)^2} = \frac{h(-i) + h'(-i)(z+i) + \cdots}{(z+i)^2}$$

$$\Downarrow$$

$$\operatorname{Res}(-i) = h'(-i) \qquad h'(z) = \frac{-ik\,e^{-ikz}}{(z-i)^2} - \frac{2e^{-ikz}}{(z-i)^3}$$

$$\Rightarrow \operatorname{Res}(-i) = \frac{-ik\,e^{-k}}{(-2i)^2} - \frac{2e^{-k}}{(-2i)^3} = \frac{(-2k-2)\,e^{-k}}{8i} = \frac{i(k+1)\,e^{-k}}{4}$$

$$\Rightarrow \tilde{f}(k) = \frac{-2\pi i}{\sqrt{2\pi}} \cdot \frac{i(k+1)\,e^{-k}}{4} = \frac{\sqrt{\pi}\,(k+1)\,e^{-k}}{2\sqrt{2}} \qquad k \geq 0$$

For $k < 0$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{(x^2+1)^2} \, dx$$

let $\tilde{x} := -x$ $\quad d\tilde{x} = -dx$

$$\Rightarrow \quad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{+\infty}^{-\infty} - \frac{e^{ik\tilde{x}}}{(\tilde{x}^2+1)^2} \, d\tilde{x}$$

let $\tilde{k} := -k \quad \Rightarrow \quad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-i\tilde{k}\tilde{x}}}{(\tilde{x}^2+1)^2} \, d\tilde{x}$

$$= \frac{\sqrt{\pi}\,(\tilde{k}+1)\, e^{-\tilde{k}}}{2\sqrt{2}}$$

$$= \frac{\sqrt{\pi}\,(-k+1)\, e^{k}}{2\sqrt{2}}$$

$\Downarrow$

$$\tilde{f}(k) = \frac{\sqrt{\pi}\,(|k|+1)\, e^{-|k|}}{2\sqrt{2}}$$