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FUNCTIONS OF A COMPLEX VARIABLE

1. INTRODUCTION

In Chapter 2 we discussed plotting complex numbers $z = x + iy$ in the complex plane (see Figure 1.1) and finding values of the elementary functions of z such as roots, trigonometric functions, logarithms, etc. Now we want to discuss the calculus of functions of z, differentiation, integration, power series, etc. As you know from such topics as differential equations, Fourier series and integrals, mechanics, electricity, etc., it is often very convenient to use complex expressions. The basic facts and theorems about functions of a complex variable not only simplify many calculations but often lead to a better understanding of a problem and consequently to a more efficient method of solution. We are going to state some of the basic definitions and theorems of the subject (omitting the longer proofs), and show some of their uses.

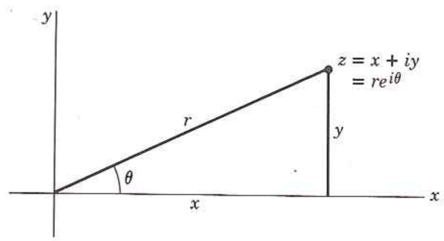


FIGURE 1.1

As we saw in Chapter 2, the value of a function of z for a given z is a complex number. Consider a simple function of z, namely $f(z) = z^2$. We may write

$$f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy = u(x, y) + iv(x, y),$$

where $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. In Chapter 2, we observed that a complex number $z = x + iy$ is equivalent to a pair of real numbers x, y. Here we may note that a function of z is equivalent to a pair of real functions, $u(x, y)$ and $v(x, y)$, of the real variables x and y. In general, we write

$$(1.1) \qquad f(z) = f(x + iy) = u(x, y) + iv(x, y),$$

where it is understood that u and v are real functions of the real variables x and y.

Recall that functions are customarily *single-valued*, that is, $f(z)$ has just one (complex) value for each z. Does this mean that we cannot define a function by a formula such as $\ln z$ or arc tan z? By Chapter 2, we have

$$\ln z = \ln |z| + i(\theta + 2n\pi),$$

where $\tan \theta = y/x$. For each z, $\ln z$ has an infinite set of values. But if θ is allowed a range of only 2π, then $\ln z$ has one value for each z and this single-valued function is called a *branch* of $\ln z$. Thus in using formulas such as $\sqrt{z}$, $\ln z$, arc tan z, to define functions, we always discuss a single branch at a time so that we have a single-valued function. (As a matter of terminology, however, you should know that the whole collection of branches is often called a "multiple-valued function.")

PROBLEMS, SECTION 1

Find the real and imaginary parts $u(x, y)$ and $v(x, y)$ of the following functions.

1. z^3
2. z
3. $\bar{z}$
4. $|z|$
5. Re z
6. e^z
7. $\cosh z$
8. $\sin z$
9. $\dfrac{1}{z}$
10. $\dfrac{2z+3}{z+2}$
11. $\dfrac{2z-i}{iz+2}$
12. $\dfrac{z}{z^2+1}$
13. $\ln |z|$
14. $z^2\bar{z}$
15. $e^{\bar{z}}$
16. $z^2 - \bar{z}^2$
17. $\cos \bar{z}$
18. $\sqrt{z}$
19. $\ln z$ (Use $0 < \theta < 2\pi$.)
20. $(1 + 2i)z^2 + (i - 1)z + 3$
21. e^{iz} (*Careful*; $\cos z$ and $\sin z$ are *not* u and v.)

2. ANALYTIC FUNCTIONS

Definition The derivative of $f(z)$ is defined (just as for a function of a real variable) by the equation

$$(2.1) \qquad f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z},$$

where

$$\Delta f = f(z + \Delta z) - f(z),$$
$$\Delta z = \Delta x + i\Delta y.$$

Definition: A function $f(z)$ is *analytic* (or *regular* or *holomorphic* or *monogenic*) in a region* of the complex plane if it has a (unique) derivative at every point of the region. The statement "$f(z)$ is analytic *at a point* $z = a$" means that $f(z)$ has a derivative at every point inside some small circle about $z = a$.

Let us consider what it means for $f(z)$ to have a derivative. First think about a function $f(x)$ of a real variable x; it is possible for the limit of $\Delta f/\Delta x$ to have two values at a point x_0, as shown in Figure 2.1—one value when we approach x_0 from the left and a different value when we approach x_0 from the right. When we say that $f(x)$ has a derivative at $x = x_0$, we mean that these two values are equal. However, for a function $f(z)$ of a complex variable z, there are an infinite number of ways we can approach a point z_0; a few ways are shown in Figure 2.2. When we say that $f(z)$ has a derivative at $z = z_0$, we mean that $f'(z)$ [as defined by (2.1)] has the same value no matter how we approach z_0. This is an amazingly stringent requirement and we might well wonder whether there *are* any analytic functions. On the other hand, it is hard to imagine making any progress in calculus unless we can find derivatives!

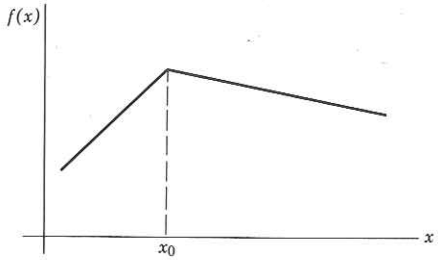


FIGURE 2.1

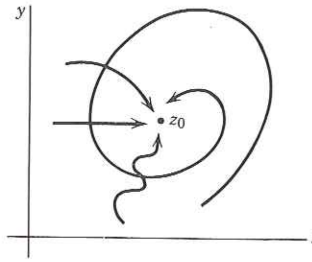


FIGURE 2.2

Let us immediately reassure ourselves that there *are* analytic functions by using the definition (2.1) to find the derivatives of some simple functions. For example, let us show that $(d/dz)(z^2) = 2z$. By (2.1) we have

$$\frac{d}{dz}(z^2) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^2 + 2z\,\Delta z + (\Delta z)^2 - z^2}{\Delta z}$$
$$= \lim_{\Delta z \to 0} (2z + \Delta z) = 2z.$$

We see that the result is independent of *how* Δz tends to zero; thus z^2 is an analytic function. By the same method it follows that $(d/dz)(z^n) = nz^{n-1}$ if n is a positive integer (Problem 30).

* Isolated points and curves are not regions; a region must be two-dimensional.

Now what we have just been doing is nothing but the familiar Δ-process! We observe that the definition (2.1) of a derivative is of exactly the same form as the corresponding definition for a function of a real variable. Because of this similarity, many familiar formulas can be proved by the same methods used in the real case, as we have just discovered in differentiating z^2. You can easily show (Problems 25 to 28) that derivatives of sums, products, and quotients follow the familiar rules and that the chain rule holds [if $f = f(g)$ and $g = g(z)$, then $df/dz = (df/dg)(dg/dz)$]. Then derivatives of rational functions of z follow the familiar real-variable formulas. If we assume the definitions and theorems of Chapters 1 and 2, we can see that the derivatives of the other elementary functions also follow the familiar formulas; for example, $(d/dz)(\sin z) = \cos z$, etc. (Problems 29 to 33).

Now you may be wondering what is new here since all our results so far seem to be just the same as for functions of a real variable. [The reason for this is that we have been discussing only functions $f(z)$ that *have* derivatives.] In Figures 2.1 and 2.2 we pointed out the essential difference between finding $(d/dx)f(x)$ and finding $(d/dz)f(z)$, namely that there are an infinite number of ways we can approach z_0 in Figure 2.2. To see an example of this let us try to find $(d/dz)(|z|^2)$. (Note that $|x|^2 = x^2$, and its derivative is $2x$.) If $|z|^2$ has a derivative, it is given by (2.1), that is, by

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{|z + \Delta z|^2 - |z|^2}{\Delta z}.$$

The numerator of this fraction is always real (because absolute values are real—recall $|z| = \sqrt{x^2 + y^2} = r$). Consider the denominator $\Delta z = \Delta x + i\Delta y$. As we approach z_0 in Figure 2.2 (that is, let $\Delta z \to 0$), Δz has different values depending on our method of approach. For example, if we come in along a horizontal line, then $\Delta y = 0$ and $\Delta z = \Delta x$; along a vertical line $\Delta x = 0$ so $\Delta z = i\Delta y$, and along other directions Δz is some complex number; in general, Δz is neither real nor pure imaginary. Since the numerator of $\Delta f/\Delta z$ is real and the denominator may be real or imaginary (in general, complex), we see that $\lim_{\Delta z \to 0} \Delta f/\Delta z$ has different values for different directions of approach to z_0, that is, $|z|^2$ is not analytic.

Now we have seen examples of both analytic and nonanalytic functions, but we still do not know how to tell whether a function has a derivative [except to appeal to (2.1)]. The following theorems answer this question.

Theorem I (which we shall prove). If $f(z) = u(x, y) + iv(x, y)$ is analytic in a region, then in that region

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{2.2}$$

These equations are called the *Cauchy-Riemann conditions.*

Proof. Remembering that $f = f(z)$, where $z = x + iy$, we find by the rules of partial differentiation (see Problem 28 and also Chapter 4)

$$\frac{\partial f}{\partial x} = \frac{df}{dz}\frac{\partial z}{\partial x} = \frac{df}{dz} \cdot 1, \qquad \frac{\partial f}{\partial y} = \frac{df}{dz}\frac{\partial z}{\partial y} = \frac{df}{dz} \cdot i. \tag{2.3}$$

Since $f = u(x, y) + iv(x, y)$ by (1.1), we also have

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}. \tag{2.4}$$

Notice that if f has a derivative with respect to z, then it also has partial derivatives with respect to x and y by (2.3). Since a complex function has a derivative with respect to a real variable if and only if its real and imaginary parts do [see (1.1)], then by (2.4) u and v also have partial derivatives with respect to x and y. Combining (2.3) and (2.4) we have

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and

$$\frac{df}{dz} = \frac{1}{i}\frac{\partial f}{\partial y} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Since we assumed that df/dz exists and is unique (this is what analytic means), these two expressions for df/dz must be equal. Taking real and imaginary parts, we get the Cauchy-Riemann equations (2.2).

Theorem II (which we state without proof). If $u(x, y)$ and $v(x, y)$ and their partial derivatives with respect to x and y are continuous and satisfy the Cauchy-Riemann conditions in a region, then $f(z)$ is analytic at all points inside the region (not necessarily on the boundary).

Although we shall not prove this (for proof see texts on complex variable) we can make it plausible by showing that it is true when we approach z_0 along any straight line. We shall calculate df/dz assuming that we approach z_0 along a straight line of slope m, and we shall show that df/dz does not depend on m if u and v satisfy (2.2). The equation of the straight line of slope m through the point $z_0 = x_0 + iy_0$ is

$$y - y_0 = m(x - x_0)$$

and along this line we have $dy/dx = m$. Then we find

$$\frac{df}{dz} = \frac{du + i\,dv}{dx + i\,dy} = \frac{\dfrac{\partial u}{\partial x}dx + \dfrac{\partial u}{\partial y}dy + i\left(\dfrac{\partial v}{\partial x}dx + \dfrac{\partial v}{\partial y}dy\right)}{dx + i\,dy}$$

$$= \frac{\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}m + i\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial v}{\partial y}m\right)}{1 + im}$$

Using the Cauchy-Riemann equations (2.2), we get

$$\frac{df}{dz} = \frac{\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial x} m + i\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial x} m\right)}{1 + im}$$

$$= \frac{\dfrac{\partial u}{\partial x}(1 + im) + i\dfrac{\partial v}{\partial x}(1 + im)}{1 + im} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Thus df/dz has the same value when calculated for approach along *any* straight line. The theorem states that it also has the same value for approach along *any curve*.

Some definitions:

A *regular point* of $f(z)$ is a point at which $f(z)$ is analytic.

A *singular point* or *singularity* of $f(z)$ is a point at which $f(z)$ is not analytic. It is called an *isolated* singular point if $f(z)$ is analytic everywhere else inside some small circle about the singular point.

Theorem III (which we state without proof). If $f(z)$ is analytic in a region (R in Figure 2.3), then it has derivatives of all orders at points inside the region and can be expanded in a Taylor series about any point z_0 inside the region. The power series converges inside the circle about z_0 that extends to the nearest singular point (C in Figure 2.3).

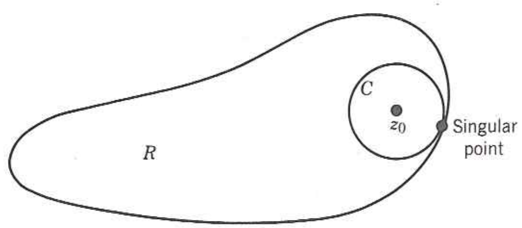


FIGURE 2.3

Notice again what a strong condition it is on $f(z)$ to say that it has a derivative. It is quite possible for a function of a real variable $f(x)$ to have a first derivative but not higher derivatives. But if $f(z)$ has a first derivative with respect to z, then it has derivatives of all orders.

This theorem also explains a fact about power series which may have puzzled you. The function $f(x) = 1/(1 + x^2)$ does not have anything peculiar about its behavior at $x = \pm 1$. Yet if we expand it in a power series

$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \cdots \tag{2.5}$$

we see that the series converges only for $|x| < 1$. We can see why this happens if we consider instead

$$f(z) = \frac{1}{1 + z^2} = 1 - z^2 + z^4 - z^6 + \cdots. \tag{2.6}$$

When $z = \pm i$, $f(z)$ and its derivatives become infinite; that is, $f(z)$ is not analytic in any region containing $z = \pm i$. The point z_0 of the theorem is the origin and the circle C (Figure 2.4) of convergence of the series extends to the nearest singular points $\pm i$. The series converges *inside* C. Since a power series in z always converges inside its circle of convergence and diverges outside (Chapter 2, Problem 6.14), we see that (2.5) [which is (2.6) for $y = 0$] can converge only for $|x| < 1$.

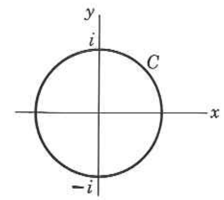


FIGURE 2.4

A function $\phi(x, y)$ which satisfies Laplace's equation, namely, $\nabla^2\phi = \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0$, is called a *harmonic* function. A great many physical problems lead to Laplace's equation, and consequently we are very much interested in finding solutions of it. (See Section 10 and Chapter 13.) The following theorem should then give you a clue as to one reason why the theory of functions of a complex variable is important in applications.

Theorem IV. Part 1 (to be proved in Problem 44). If $f(z) = u + iv$ is analytic in a region, then u and v satisfy Laplace's equation in the region (that is, u and v are harmonic functions).

Part 2 (which we state without proof). Any function u (or v) satisfying Laplace's equation in a simply-connected region, is the real or imaginary part of an analytic function $f(z)$.

Thus we can find solutions of Laplace's equation simply by taking the real or imaginary parts of an analytic function of z. It is also often possible, starting with a simple function which satisfies Laplace's equation, to find the explicit function $f(z)$ of which it is, say, the real part.

Example. Consider the function $u(x, y) = x^2 - y^2$. We find that

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0,$$

that is, u satisfies Laplace's equation (or u is a harmonic function). Let us find the function $v(x, y)$ such that $u + iv$ is an analytic function of z. By the Cauchy-Riemann equations

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x.$$

Integrating partially with respect to y, we get

$$v(x, y) = 2xy + g(x),$$

where $g(x)$ is a function of x to be found. Differentiating partially with respect to x and again using the Cauchy-Riemann equations, we have

$$\frac{\partial v}{\partial x} = 2y + g'(x) = -\frac{\partial u}{\partial y} = 2y.$$

Thus we find

$$g'(x) = 0, \qquad \text{or} \quad g = \text{const.}$$

Then

$$f(z) = u + iv = x^2 - y^2 + 2ixy + \text{const.} = z^2 + \text{const.}$$

The pair of functions u, v are called *conjugate harmonic functions.* (Also see Problem 64.)

PROBLEMS, SECTION 2

1 to 21. Use the Cauchy-Riemann conditions to find out whether the functions in Problems 1.1 to 1.21 are analytic. Similarly, find out whether the following functions are analytic.

22. $y + ix$ 23. $\dfrac{x - iy}{x^2 + y^2}$ 24. $\dfrac{y - ix}{x^2 + y^2}$

Using the definition (2.1) of $(d/dz)f(z)$, show that the following familiar formulas hold. *Hint:* Use the same methods as for functions of a real variable.

25. $\dfrac{d}{dz}[Af(z) + Bg(z)] = A\dfrac{df}{dz} + B\dfrac{dg}{dz}.$ 26. $\dfrac{d}{dz}[f(z)g(z)] = f(z)\dfrac{dg}{dz} + g(z)\dfrac{df}{dz}.$

27. $\dfrac{d}{dz}\left(\dfrac{f(z)}{g(z)}\right) = \dfrac{gf' - fg'}{g^2}, \quad g(z) \neq 0.$ 28. $\dfrac{d}{dz}f[g(z)] = \dfrac{df}{dg}\dfrac{dg}{dz}.$ (See hint below.)

Problem 28 is the chain rule for the derivative of a function of a function. *Hint:* Assume that df/dg and dg/dz exist, and write equations like (3.5) of Chapter 4 for Δf and Δg; substitute Δg into Δf, divide by Δz, and take limits.

29. $\dfrac{d}{dz}(z^3) = 3z^2.$ 30. $\dfrac{d}{dz}(z^n) = nz^{n-1}.$

31. $\dfrac{d}{dz}\ln z = \dfrac{1}{z}, \quad z \neq 0.$ *Hint:* Expand $\ln\left(1 + \dfrac{\Delta z}{z}\right)$ in series.

32. Using the definition of e^z by its power series [(8.1) of Chapter 2], and the theorem (Chapters 1 and 2) that power series may be differentiated term by term (within the circle of convergence), and the result of Problem 30, show that $(d/dz)(e^z) = e^z$.

33. Using the definitions of $\sin z$ and $\cos z$ [Chapter 2, equation (11.4)], find their derivatives. Then using Problem 27, find $(d/dz)(\cot z)$, $z \neq n\pi$.

Using series you know from Chapter 1, write the power series (about the origin) of the following functions. Use Theorem III to find the circle of convergence of each series. What you are looking for is the point (anywhere in the complex plane) nearest the origin, at which the function

does not have a derivative. Then the circle of convergence has center at the origin and extends to that point. The series converges *inside* the circle.

34. $\ln(1 - z)$ 35. $\cos z$ 36. $\sqrt{1 + z^2}$

37. $\tanh z$ 38. $\dfrac{1}{2i + z}$ 39. $\dfrac{z}{z^2 + 9}$

40. $(1 - z)^{-1}$ 41. e^{iz} 42. $\sinh z$

43. In Chapter 12, equations (5.1) and (5.2), we expanded the function $\Phi(x, h)$ in a series of powers of h. Use Theorem III (see instructions for Problems 34 to 42 above) to show that the series for $\Phi(x, h)$ converges for $|h| < 1$ and $-1 \le x \le 1$. Here h is the variable and x is a parameter; you should find the (complex) value of h which makes Φ infinite, and show that the absolute value of this complex number is 1 (independent of x when $x^2 \le 1$). This proves that the series for real h converges for $|h| < 1$.

44. Prove Theorem IV, Part 1. *Hint:* Recall the equality of the second cross partial derivatives; see Chapter 4, end of Section 1.

45. Let $f(z) = u + iv$ be an analytic function, and let $\mathbf{F}$ be the vector $\mathbf{F} = v\mathbf{i} + u\mathbf{j}$. Show that the equations div $\mathbf{F} = 0$ and curl $\mathbf{F} = 0$ are equivalent to the Cauchy-Riemann equations.

46. Find the Cauchy-Riemann equations in polar coordinates. *Hint:* $z = re^{i\theta}$, $f(z) = u(r, \theta) + iv(r, \theta)$. Follow the method of equations (2.3) and (2.4).

47. Using your results in Problem 46 and the method of Problem 44, show that u and v satisfy Laplace's equation in polar coordinates (see Chapter 10, Section 9) if $f(z) = u + iv$ is analytic.

Using polar coordinates (Problem 46), find out whether the following functions satisfy the Cauchy-Riemann equations.

48. $\sqrt{z}$ 49. $|z|$ 50. $\ln z$

51. z^n 52. $|z|^2$ 53. $|z|^{1/2}e^{i\theta/2}$

Show that the following functions are harmonic, that is, that they satisfy Laplace's equation, and find for each a function $f(z)$ of which the given function is the real part. Show that the function $v(x, y)$ which you find also satisfies Laplace's equation.

54. y 55. $3x^2y - y^3$ 56. xy 57. $x + y$

58. $\cosh y \cos x$ 59. $e^x \cos y$ 60. $\ln(x^2 + y^2)$

61. $\dfrac{x}{x^2 + y^2}$ 62. $e^{-y}\sin x$ 63. $\dfrac{y}{(1 - x)^2 + y^2}$

64. It can be shown that, if $u(x, y)$ is a harmonic function which is defined at $z_0 = x_0 + iy_0$, then an analytic function of which $u(x, y)$ is the real part is given by

$$f(z) = 2u\left(\frac{z + \bar{z}_0}{2}, \frac{z - \bar{z}_0}{2i}\right) + \text{const.}$$

[See Struble, Quart. Appl. Math., 37 (1979), 79–81.] Use this formula to find $f(z)$ in Problems 54 to 63. *Hint:* If $u(0, 0)$ is defined, take $z_0 = 0$.

3. CONTOUR INTEGRALS

Theorem V. Cauchy's theorem (which we shall prove). Let C be a simple* closed curve with a continuously turning tangent except possibly at a finite number of points (that is, we allow a finite number of corners, but otherwise the curve must be "smooth"). If $f(z)$ is analytic on and inside C, then

$$\oint_{\text{around } C} f(z)\,dz = 0. \tag{3.1}$$

(This is a line integral as in vector analysis; it is called a *contour integral* in the theory of complex variables.)

Proof.

$$\oint_C f(z)\,dz = \oint_C (u + iv)(dx + i\,dy) \tag{3.2}$$
$$= \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy).$$

Green's theorem in the plane (see Chapter 6, Section 9) says that if $P(x, y)$ and $Q(x, y)$ and their partial derivatives are continuous in a simply-connected region R, then

$$\oint_C P\,dx + Q\,dy = \iint_{\substack{\text{area}\\ \text{inside } C}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy, \tag{3.3}$$

where C is a simple closed curve in R. The curve C is traversed in a direction so that the area inclosed is always to the left. The area integral is over the area inside C, with C and the area entirely in R. Applying (3.3) to the first integral in (3.2), we get

$$\oint_C (u\,dx - v\,dy) = \iint_{\substack{\text{area}\\ \text{inside } C}} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy. \tag{3.4}$$

Since $f(z)$ is analytic, u and v and their derivatives are continuous; by the Cauchy-Riemann equations the integrand on the right of (3.4) is zero at every point of the area of integration, so the integral is equal to zero. In the same way the second integral in (3.2) is zero; thus (3.1) is proved.

* A simple curve is one which does not cross itself.

Theorem VI. Cauchy's integral formula (which we shall prove). If $f(z)$ is analytic on and inside a simple closed curve C, the value of $f(z)$ at a point $z = a$ inside C is given by the following contour integral along C:

$$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z - a}\,dz.$$

Proof. Let a be a fixed point inside the simple closed curve C and consider the function

$$\phi(z) = \frac{f(z)}{z - a}, \tag{3.5}$$

where $f(z)$ is analytic on and inside C. Let C' be a small circle (inside C) with center at a and radius ρ. Make a cut between C and C' along AB (Figure 3.1); two cuts are shown to make the picture clear, but later we shall make them coincide. We are now going to integrate along the path shown in Figure 3.1 (in the direction shown by the arrows) from A, around C, to B, around C', and back to A. Notice that the area between the curves C and C' is always to the left of the path of integration and is inclosed by it. In this area between C and C', the function $\phi(z)$ is analytic; we have cut out a small disk about the point $z = a$ at which $\phi(z)$ is not analytic. Cauchy's theorem then applies to the integral along the combined path consisting of C counterclockwise, C' clockwise, and the two cuts. The two integrals, in opposite directions along the cuts, cancel when the cuts are made to coincide. Thus we have

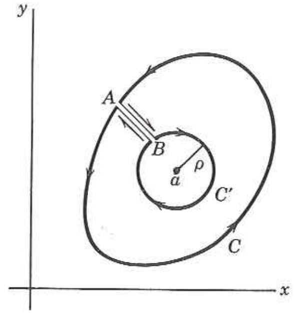


FIGURE 3.1

$$\oint_{\substack{C\text{ counter-}\\ \text{clockwise}}} \phi(z)\,dz + \oint_{C'\text{ clockwise}} \phi(z)\,dz = 0 \quad \text{or} \tag{3.6}$$

$$\oint_C \phi(z)\,dz = \oint_{C'} \phi(z)\,dz \qquad \text{where both are counterclockwise.}$$

Along the circle C', $z = a + \rho e^{i\theta}$, $dz = \rho i e^{i\theta}\,d\theta$, and (3.6) becomes

$$\oint_C \phi(z)\,dz = \oint_{C'} \phi(z)\,dz = \oint_{C'} \frac{f(z)}{z - a}\,dz \tag{3.7}$$
$$= \int_0^{2\pi} \frac{f(z)}{\rho e^{i\theta}}\,\rho i e^{i\theta}\,d\theta = \int_0^{2\pi} f(z) i\,d\theta.$$

Since our calculation is valid for any (sufficiently small) value of ρ, we shall let $\rho \to 0$ (that is, $z \to a$) to simplify the formula. Because $f(z)$ is continuous at $z = a$ (it is analytic inside C), $\lim_{z \to a} f(z) = f(a)$. Then (3.7) becomes

$$\oint_C \phi(z)\, dz = \oint_C \frac{f(z)}{z-a}\, dz = \int_0^{2\pi} f(z) i\, d\theta = \int_0^{2\pi} f(a) i\, d\theta = 2\pi i f(a) \tag{3.8}$$

or

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz, \qquad a \text{ inside } C. \tag{3.9}$$

This is Cauchy's integral formula. Note carefully that the point a is inside C; if a were outside C, then $\phi(z)$ would be analytic everywhere inside C and the integral would be zero by Cauchy's theorem. A useful way to look at (3.9) is this: If the values of $f(z)$ are given on the boundary of a region (curve C), then (3.9) gives the value of $f(z)$ at any point a inside C. With this interpretation you will find Cauchy's integral formula written with a replaced by z, and z replaced by some different dummy integration variable, say w:

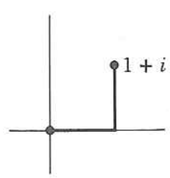
$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w-z}\, dw, \qquad z \text{ inside } C. \tag{3.10}$$

For some important uses of this theorem, see Problems 11.3 and 11.36 to 11.38.

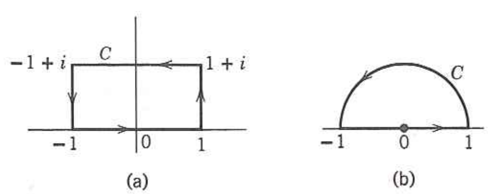
PROBLEMS, SECTION 3

Evaluate the following line integrals in the complex plane by direct integration, that is, as in Chapter 6, Section 8, *not* using theorems from this chapter. (If you see that a theorem applies, use it to check your result.)

1. $\int_i^{1+i} z\, dz$ along a straight line parallel to the x axis.
2. $\int_0^{1+i} (z^2 - z)\, dz$
   (a) along the line $y = x$;
   (b) along the indicated broken line.

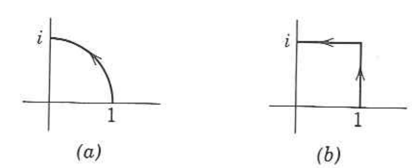


3. $\oint_C z^2\, dz$ along the indicated paths:



4. $\int dz/(1 - z^2)$ along the whole positive imaginary axis, that is, the y axis; this is frequently written as $\int_0^{i\infty} dz/(1 - z^2)$.
5. $\int e^{-z}\, dz$ along the positive part of the line $y = \pi$; this is frequently written as $\int_{i\pi}^{\infty + i\pi} e^{-z}\, dz$.

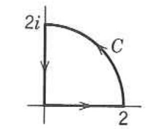
6. $\int_1^i z\, dz$ along the indicated paths:



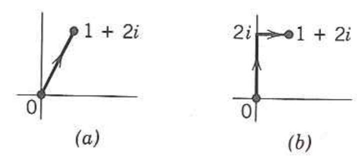
7. $\displaystyle\int \frac{dz}{8i + z^2}$ along the line $y = x$ from 0 to ∞.

8. $\displaystyle\int_{2\pi}^{2\pi + i\infty} e^{2iz}\, dz$ 9. $\displaystyle\int_{1+2i}^{\infty + 2i} \frac{dz}{(z - 2i)^2}$ 10. $\displaystyle\int_2^{2 + i\infty} z e^{iz}\, dz$

11. Evaluate $\oint_C (\bar{z} - 3)\, dz$ where C is the indicated closed curve along the first quadrant part of the circle $|z| = 2$, and the indicated parts of the x and y axes. *Hint:* Don't try to use Cauchy's theorem! (Why not? *Further hint:* See Problem 2.3.)



12. $\int_0^{1+2i} |z|^2\, dz$ along the indicated paths:



13. In Chapter 6, Section 11, we showed that a necessary condition for $\int_a^b \mathbf{F} \cdot d\mathbf{r}$ to be independent of the path of integration, that is, for $\oint_C \mathbf{F} \cdot d\mathbf{r}$ around a simple closed curve C to be zero, was curl $\mathbf{F} = 0$, or in two dimensions, $\partial F_y/\partial x = \partial F_x/\partial y$. By considering (3.2), show that the corresponding condition for $\oint_C f(z)\, dz$ to be zero is that the Cauchy-Riemann conditions hold.

14. In finding complex Fourier series in Chapter 7, we showed that

$$\int_0^{2\pi} e^{inx} e^{-imx}\, dx = 0, \qquad n \neq m.$$

Show this by applying Cauchy's theorem to

$$\oint_C z^{n-m-1}\, dz, \qquad n > m,$$

where C is the circle $|z| = 1$. (Note that although we take $n > m$ to make z^{n-m-1} analytic at $z = 0$, an identical proof using z^{m-n-1} with $n < m$ completes the proof for all $n \neq m$.)

15. If $f(z)$ is analytic on and inside the circle $|z| = 1$, show that $\displaystyle\int_0^{2\pi} e^{i\theta} f(e^{i\theta})\, d\theta = 0$.

16. If $f(z)$ is analytic in the disk $|z| \le 2$, evaluate $\int_0^{2\pi} e^{2i\theta} f(e^{i\theta})\, d\theta$.

Use Cauchy's theorem or integral formula to evaluate the integrals in Problems 17 to 20.

17. $\displaystyle\oint_C \frac{\sin z\, dz}{2z - \pi}$ where (a) C is the circle $|z| = 1$, (b) C is the circle $|z| = 2$.

18. $\displaystyle\oint_C \frac{\sin 2z\, dz}{6z - \pi}$ where C is the circle $|z| = 3$.

19. $\oint_C \frac{e^{3z}\,dz}{z - \ln 2}$ if C is the square with vertices $\pm 1 \pm i$.

20. $\oint_C \frac{\cosh z\,dz}{2\ln 2 - z}$ if C is the circle (a) $|z| = 1$; (b) $|z| = 2$.

21. Differentiate Cauchy's formula (3.9) or (3.10) to get

$$f'(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)\,dw}{(w-z)^2} \quad \text{or} \quad f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^2}.$$

By differentiating n times, obtain

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{f(w)\,dw}{(w-z)^{n+1}} \quad \text{or} \quad f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^{n+1}}.$$

Use Problem 21 to evaluate the following integrals.

22. $\oint_C \frac{\sin 2z\,dz}{(6z - \pi)^3}$ where C is the circle $|z| = 3$.

23. $\oint_C \frac{e^{3z}\,dz}{(z - \ln 2)^4}$ where C is the square in Problem 19.

24. $\oint_C \frac{\cosh z\,dz}{(2\ln 2 - z)^5}$ where C is the circle $|z| = 2$.

4. LAURENT SERIES

Theorem VII. Laurent's theorem [equation (4.1)] (which we shall state without proof). Let C_1 and C_2 be two circles with center at z_0. Let $f(z)$ be analytic in the region R between the circles. Then $f(z)$ can be expanded in a series of the form

$$\text{(4.1)} \quad f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$$

convergent in R.

Such a series is called a *Laurent series*. The "b" series in (4.1) is called the *principal part* of the Laurent series.

Example 1. Consider the Laurent series

$$\text{(4.2)} \quad f(z) = 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots + \left(\frac{z}{2}\right)^n + \cdots$$
$$+ \frac{2}{z} + 4\left(\frac{1}{z^2} - \frac{1}{z^3} + \cdots + \frac{(-1)^n}{z^n} + \cdots\right).$$

Let us see where this series converges. First consider the series of positive powers; by the ratio test (see Chapters 1 and 2), this series converges for $|z/2| < 1$, that is, for $|z| < 2$. Similarly, the series of negative powers converges for $|1/z| < 1$, that is,

$|z| > 1$. Then both series converge (and so the Laurent series converges) for $|z|$ between 1 and 2, that is, in a ring between two circles of radii 1 and 2.

We expect this result in general. The "a" series is a power series, and a power series converges *inside* some circle (say C_2 in Figure 4.1). The "b" series is a series of inverse powers of z, and so converges for $|1/z| <$ some constant; thus the "b" series converges *outside* some circle (say C_1 in Figure 4.1). Then a Laurent series converges between two circles (if it converges at all). (Note that the inner circle may be a point and the outer circle may have infinite radius).

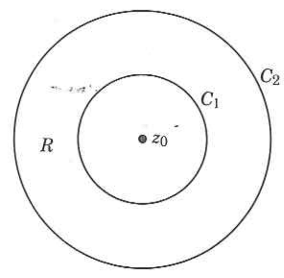


FIGURE 4.1

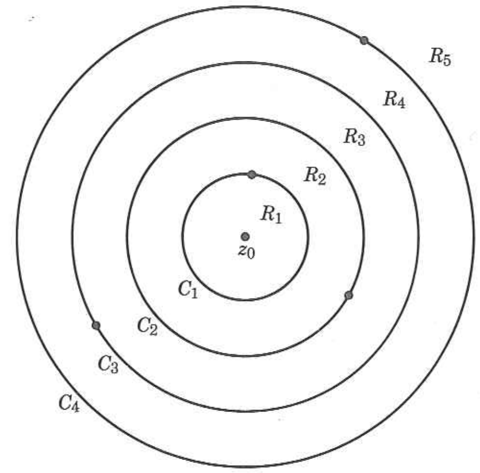


FIGURE 4.2

The formulas for the coefficients in (4.1) are (Problem 5.2)

$$\text{(4.3)} \quad a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}, \qquad b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z - z_0)^{-n+1}},$$

where C is any simple closed curve surrounding z_0 and lying in R. However, this is not usually the easiest way to find a Laurent series. Like power series about a point, the Laurent series (about z_0) for a function in a given annular ring (about z_0) where the function is analytic, is unique, and we can find it by any method we choose. (See examples below.) *Warning:* If $f(z)$ has several isolated singularities (Figure 4.2), there are several annular rings, R_1, R_2, ..., in which $f(z)$ is analytic; then there are several different Laurent series for $f(z)$, one for each ring. The Laurent series which we usually want is the one that converges near z_0. If you have any doubt about the ring of convergence of a Laurent series, you can find out by testing the "a" series and the "b" series separately.

Example 2. The function from which we obtained (4.2) was

$$\text{(4.4)} \quad f(z) = \frac{12}{z(2 - z)(1 + z)}.$$

This function has three singular points, at $z = 0$, $z = 2$, and $z = -1$. Thus there are two circles C_1 and C_2 about $z_0 = 0$ in Figure 4.2, and three Laurent series about $z_0 = 0$, one series valid in each of the three regions R_1 $(0 < |z| < 1)$, R_2 $(1 < |z| < 2)$, and R_3 $(|z| > 2)$. To find these series, we first write $f(z)$ in the following form using partial fractions (Problem 2):

$$f(z) = \frac{4}{z}\left(\frac{1}{1+z} + \frac{1}{2-z}\right). \tag{4.5}$$

Now, for $0 < |z| < 1$, we expand each of the fractions in the parenthesis in (4.5) in powers of z. This gives (Problem 2):

$$f(z) = -3 + 9z/2 - 15z^2/4 + 33z^3/8 + \cdots + 6/z. \tag{4.6}$$

This is the Laurent series for $f(z)$ which is valid in the region $0 < |z| < 1$. To obtain the series valid in the region $|z| > 2$, we write the fractions in (4.5) as

$$\frac{1}{1+z} = \frac{1}{z}\frac{1}{1+1/z}, \qquad \frac{1}{2-z} = -\frac{1}{z}\frac{1}{1-2/z} \tag{4.7}$$

and expand each fraction in powers of $1/z$. This gives the Laurent series valid for $|z| > 2$ (Problem 2):

$$f(z) = -(12/z^3)(1 + 1/z + 3/z^2 + 5/z^3 + 11/z^4 + \cdots). \tag{4.8}$$

Finally, to obtain (4.2), we expand the fraction $1/(2-z)$ in powers of z, and the fraction $1/(1+z)$ in powers of $1/z$; this gives a Laurent series which converges for $1 < |z| < 2$. Thus the Laurent series (4.6), (4.2), and (4.8) all represent $f(z)$ in (4.4), but in three different regions.

Let z_0 in Figure 4.2 be either a regular point or an isolated singular point and assume that there are no other singular points inside C_1. Let $f(z)$ be expanded in the Laurent series about z_0 which converges inside C_1 (except possibly at z_0); we say that we have expanded $f(z)$ in the Laurent series which converges near z_0. Then we have the following definitions.

Definitions:

If all the b's are zero, $f(z)$ is analytic at $z = z_0$, and we call z_0 a *regular point*. (See Problem 4.1.)

If $b_n \neq 0$, but all the b's after b_n are zero, $f(z)$ is said to have a *pole of order n* at $z = z_0$. If $n = 1$, we say that $f(z)$ has a *simple pole*.

If there are an infinite number of b's different from zero, $f(z)$ has an *essential singularity* at $z = z_0$.

The coefficient b_1 of $1/(z - z_0)$ is called the *residue* of $f(z)$ at $z = z_0$.

Example 3.

(a) $$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

is analytic at $z = 0$; the residue of e^z at $z = 0$ is 0.

(b) $$\frac{e^z}{z^3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!z} + \frac{1}{3!} + \cdots$$

has a pole of order 3 at $z = 0$; the residue of e^z/z^3 at $z = 0$ is $1/2!$.

(c) $$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$$

has an essential singularity at $z = 0$; the residue of $e^{1/z}$ at $z = 0$ is 1.

Most of the functions we shall consider will be analytic except for poles—such functions are called *meromorphic* functions. If $f(z)$ has a pole at $z = z_0$, then $|f(z)| \to \infty$ as $z \to z_0$. A three-dimensional graph with $|f(z)|$ plotted vertically over a horizontal complex plane would look like a tapered pole near $z = z_0$. We can often see that a function has a pole and find the order of the pole without finding the Laurent series.

Example 4.

(a) $$\frac{z+3}{z^2(z-1)^3(z+1)}$$

has a pole of order 2 at $z = 0$, a pole of order 3 at $z = 1$, and a simple pole at $z = -1$.

(b) $\dfrac{\sin^2 z}{z^3}$ has a simple pole at $z = 0$.

To see that these results are correct, consider finding the Laurent series for $f(z) = g(z)/(z - z_0)^n$. We write $g(z) = a_0 + a_1(z - z_0) + \cdots$; then the Laurent series for $f(z)$ starts with the term $(z - z_0)^{-n}$ unless $a_0 = 0$, that is unless $g(z_0) = 0$. Then the order of the pole of $f(z)$ is n unless some factors cancel. In Example 4b, the sin z series starts with z, so $\sin^2 z$ has a factor z^2; thus $(\sin^2 z)/z^3$ has a simple pole at $z = 0$.

PROBLEMS, SECTION 4

1. Show that the sum of a power series which converges in a circle C is an analytic function inside C. *Hint*: See Chapter 2, Section 7, and Chapter 1, Section 11, and the definition of an analytic function.
2. Show that equation (4.4) can be written as (4.5). Then expand each of the fractions in the parenthesis in (4.5) in powers of z and in powers of $1/z$ [see equation (4.7)] and combine the series to obtain (4.6), (4.8), and (4.2).

For each of the following functions find the first few terms of each of the Laurent series about the origin, that is, one series for each annular ring between singular points. Find the residue of each function at the origin. (*Warning:* To find the residue, you must use the Laurent series which converges near the origin.) *Hints:* See Problem 2. Use partial fractions as in equations (4.5) and (4.7). Expand a term $1/(z - a)$ in powers of z to get a series convergent for $|z| < a$, and in powers of $1/z$ to get a series convergent for $|z| > a$.

3. $\dfrac{1}{z(z-1)(z-2)}$ 4. $\dfrac{1}{z(z-1)(z-2)^2}$ 5. $\dfrac{z-1}{z^3(z-2)}$

6. $\dfrac{1}{z^2(1+z)^2}$ 7. $\dfrac{2-z}{1-z^2}$ 8. $\dfrac{30}{(1+z)(z-2)(3+z)}$

For each of the following functions, say whether the indicated point is regular, an essential singularity, or a pole, and if a pole of what order it is.

9. (a) $\dfrac{\sin z}{z}$, $z=0$ (b) $\dfrac{\cos z}{z^3}$, $z=0$

(c) $\dfrac{z^3-1}{(z-1)^3}$, $z=1$ (d) $\dfrac{e^z}{z-1}$, $z=1$

10. (a) $\dfrac{e^z-1}{z^2+4}$, $z=2i$ (b) $\tan^2 z$, $z=\pi/2$

(c) $\dfrac{1-\cos z}{z^4}$, $z=0$ (d) $\cos\left(\dfrac{\pi}{z-\pi}\right)$, $z=\pi$

11. (a) $\dfrac{e^z-1-z}{z^2}$, $z=0$ (b) $\dfrac{\sin z}{z^3}$, $z=0$

(c) $\dfrac{z^2-1}{(z-1)^2}$, $z=1$ (d) $\dfrac{\cos z}{(z-\pi/2)^4}$, $z=\pi/2$

12. (a) $\dfrac{\sin z - z}{z^6}$, $z=0$ (b) $\dfrac{z^2-1}{(z^2+1)^2}$, $z=i$

(c) $ze^{1/z}$, $z=0$ (d) $\Gamma(z)$, $z=0$ [See Chapter 11, equation (4.1).]

5. THE RESIDUE THEOREM

Let z_0 be an isolated singular point of $f(z)$. We are going to find the value of $\oint_C f(z)\,dz$ around a simple closed curve C surrounding z_0 but inclosing no other singularities. Let $f(z)$ be expanded in the Laurent series (4.1) about $z=z_0$ that converges near $z=z_0$. By Cauchy's theorem (V), the integral of the "a" series is zero since this part is analytic. To evaluate the integrals of the terms in the "b" series in (4.1), we replace the integrals around C by integrals around a circle C' with center at z_0 and radius ρ as in (3.6), (3.7), and Figure 3.1. Along C', $z=z_0+\rho e^{i\theta}$; calculating the integral of the b_1 term in (4.1), we find

$$\oint_C \frac{b_1\,dz}{(z-z_0)} = b_1\int_0^{2\pi}\frac{\rho i e^{i\theta}\,d\theta}{\rho e^{i\theta}} = 2\pi i b_1. \tag{5.1}$$

It is straightforward to show (Problem 1) that the integrals of all the other b_n terms are zero. Then $\oint_C f(z)\,dz = 2\pi i b_1$, or since b_1 is called the residue of $f(z)$ at $z=z_0$, we can say

$$\oint_C f(z)\,dz = 2\pi i\cdot\text{residue of } f(z) \text{ at the singular point inside } C.$$

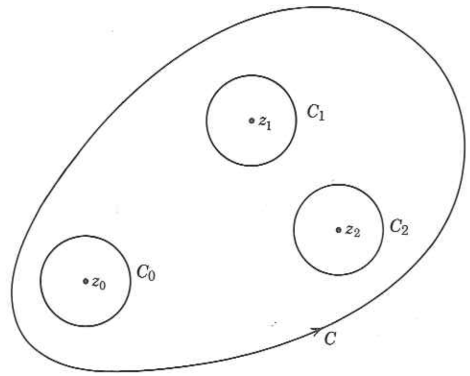


FIGURE 5.1

The only term of the Laurent series which has survived the integration process is the b_1 term; you can see the reason for the term "residue." If there are several isolated singularities inside C, say at z_0, z_1, z_2, $\cdots$, we draw small circles about each as shown in Figure 5.1 so that $f(z)$ is analytic in the region between C and the circles. Then, introducing cuts as in the proof of Cauchy's integral formula, we find that the integral around C counterclockwise, plus the integrals around the circles clockwise, is zero (since the integrals along the cuts cancel), or the integral along C is the sum of the integrals around the circles (all counterclockwise). But by (5.1), the integral around each circle is $2\pi i$ times the residue of $f(z)$ at the singular point inside. Thus we have the *residue theorem*:

$$\oint_C f(z)\,dz = 2\pi i\cdot\text{sum of the residues of } f(z) \text{ inside } C, \tag{5.2}$$

where the integral around C is in the counterclockwise direction.

The residue theorem is very useful in evaluating many definite integrals; we shall consider this in Section 7. But first, in Section 6, we need to develop some techniques for finding residues.

PROBLEMS, SECTION 5

1. If C is a circle of radius ρ about z_0, show that

$$\oint_C \frac{dz}{(z-z_0)^n} = 2\pi i \quad \text{if} \quad n=1,$$

but for any other integral value of n, positive or negative, the integral is zero. *Hint:* Use the fact that $z=z_0+\rho e^{i\theta}$ on C.

2. Verify the formulas (4.3) for the coefficients in a Laurent series. *Hint:* To get a_n, divide equation (4.1) by $(z-z_0)^{n+1}$ and use the results of Problem 1 to evaluate the integrals of the terms of the series. Use a similar method to find b_n.

3. Obtain Cauchy's integral formula (3.9) from the residue theorem (5.2).

6. METHODS OF FINDING RESIDUES

A. Laurent Series If it is easy to write down the Laurent series for $f(z)$ about $z = z_0$ that is valid near z_0, then the residue is just the coefficient b_1 of the term $1/(z - z_0)$. *Caution:* Be sure you have the expansion about $z = z_0$; the series you have memorized for e^z, $\sin z$, etc., are expansions about $z = 0$ and so can be used only for finding residues at the origin (see Section 4, Example 3). Here is another example: Given $f(z) = e^z/(z-1)$, find the residue, $R(1)$, of $f(z)$ at $z = 1$. We want to expand e^z in powers of $z - 1$; we write

$$\frac{e^z}{z-1} = \frac{e \cdot e^{z-1}}{z-1} = \frac{e}{z-1}\left[1 + (z-1) + \frac{(z-1)^2}{2!} + \cdots\right]$$

$$= \frac{e}{z-1} + e + \cdots.$$

Then the residue is the coefficient of $1/(z-1)$, that is,

$$R(1) = e.$$

B. Simple Pole If $f(z)$ has a simple pole at $z = z_0$, we find the residue by multiplying $f(z)$ by $(z - z_0)$ and evaluating the result at $z = z_0$ (Problem 10).

Example 1. Find $R(-\frac{1}{2})$ and $R(5)$ for

$$f(z) = \frac{z}{(2z+1)(5-z)}.$$

Multiply $f(z)$ by $(z + \frac{1}{2})$ [*Caution:* **not** by $(2z + 1)$] and evaluate the result at $z = -\frac{1}{2}$. We find

$$(z + \tfrac{1}{2})f(z) = (z + \tfrac{1}{2})\frac{z}{(2z+1)(5-z)} = \frac{z}{2(5-z)},$$

$$R(-\tfrac{1}{2}) = \frac{-\frac{1}{2}}{2(5+\frac{1}{2})} = -\tfrac{1}{22}.$$

Similarly,

$$(z-5)f(z) = (z-5)\frac{z}{(2z+1)(5-z)} = -\frac{z}{2z+1},$$

$$R(5) = -\tfrac{5}{11}.$$

Example 2. Find $R(0)$ for $f(z) = (\cos z)/z$.

Since $zf(z) = \cos z$, we have

$$R(0) = (\cos z)_{z=0} = \cos 0 = 1.$$

To use this method, we may in some problems have to evaluate an indeterminate form, so in general we write

(6.1) $$R(z_0) = \lim_{z \to z_0} (z - z_0) f(z) \qquad \text{when } z_0 \text{ is a simple pole.}$$

Example 3. Find the residue of $\cot z$ at $z = 0$.

By (6.1),

$$R(0) = \lim_{z\to 0} \frac{z \cos z}{\sin z} = \cos 0 \cdot \lim_{z \to 0} \frac{z}{\sin z} = 1 \cdot 1 = 1.$$

If, as often happens, $f(z)$ can be written as $g(z)/h(z)$, where $g(z)$ is analytic and not zero at z_0 and $h(z_0) = 0$, then (6.1) becomes

$$R(z_0) = \lim_{z\to z_0} \frac{(z-z_0)g(z)}{h(z)} = g(z_0) \lim_{z \to z_0} \frac{z - z_0}{h(z)} = g(z_0) \lim_{z\to z_0} \frac{1}{h'(z)} = \frac{g(z_0)}{h'(z_0)}$$

by L'Hôpital's rule or the definition of $h'(z)$ (Problem 11).

Thus we have

(6.2) $$R(z_0) = \frac{g(z_0)}{h'(z_0)} \quad \text{if} \quad \begin{cases} f(z) = g(z)/h(z), \text{ and} \\ g(z_0) = \text{finite const.} \neq 0, \text{ and} \\ h(z_0) = 0,\ h'(z_0) \neq 0. \end{cases}$$

Often (6.2) gives the most convenient way of finding the residue at a simple pole.

Example 4. Find the residue of $(\sin z)/(1 - z^4)$ at $z = i$.

By (6.2) we have

$$R(i) = \left.\frac{\sin z}{-4z^3}\right|_{z=i} = \frac{\sin i}{-4i^3} = \frac{e^{-1} - e}{(2i)(4i)} = \tfrac{1}{8}(e - e^{-1}) = \tfrac{1}{4}\sinh 1.$$

Now you may ask how you know, without finding the Laurent series, that a function has a simple pole. Perhaps the simplest answer is that if the limit obtained using (6.1) is some constant (not 0 or ∞), then $f(z)$ *does* have a simple pole and the constant is the residue. [If the limit $= 0$, the function is analytic and the residue $= 0$; if the limit is infinite, the pole is of higher order.] However, you can often recognize the order of a pole in advance. [See end of Section 4 for the simple case in which $(z - z_0)^n$ is a factor of the denominator.] Suppose $f(z)$ is written in the form $g(z)/h(z)$, where $g(z)$ and $h(z)$ are analytic. Then you can think of $g(z)$ and $h(z)$ as power series in $(z - z_0)$. If the denominator has the factor $(z - z_0)$ to *one* higher power than the numerator, then $f(z)$ has a simple pole at z_0. For example,

$$z \cot^2 z = \frac{z \cos^2 z}{\sin^2 z} = \frac{z(1 - z^2/2 + \cdots)^2}{(z - z^3/3! + \cdots)^2} = \frac{z(1 + \cdots)}{z^2(1 + \cdots)}$$

has a simple pole at $z = 0$. By the same method we can see whether a function has a pole of any order.

C. Multiple Poles When $f(z)$ has a pole of order n, we can use the following method of finding residues.

> Multiply $f(z)$ by $(z - z_0)^m$, where m is an integer greater than or equal to the order n of the pole, differentiate the result $m - 1$ times, divide by $(m - 1)!$, and evaluate the resulting expression at $z = z_0$.

It is easy to prove that this rule is correct (Problem 12) by using the Laurent series (4.1) for $f(z)$ and showing that the result of the outlined process is b_1.

Example 5. Find the residue of $f(z) = (z \sin z)/(z - \pi)^3$ at $z = \pi$.

We take $m = 3$ to eliminate the denominator before differentiating; this is an allowed choice for m because the order of the pole of $f(z)$ at π is not greater than 3 since $z \sin z$ is finite at π. (The pole is actually of order 2, but we do not need this fact.) Then following the rule stated, we get

$$R(\pi) = \frac{1}{2!}\frac{d^2}{dz^2}(z \sin z)\bigg|_{z=\pi} = \tfrac{1}{2}[-z \sin z + 2 \cos z]_{z=\pi} = -1.$$

(To compute the derivative quickly, use Leibniz' rule for differentiating a product; see Chapter 12, Section 3.)

PROBLEMS, SECTION 6

Find the Laurent series for the following functions about the indicated points; hence find the residue of the function at the point. (Be sure you have the Laurent series which converges near the point.)

1. $\dfrac{1}{z(z+1)}, \quad z = 0$
2. $\dfrac{1}{z(z-1)}, \quad z = 1$
3. $\dfrac{\sin z}{z^4}, \quad z = 0$
4. $\dfrac{\cosh z}{z^2}, \quad z = 0$
5. $\dfrac{e^z}{z^2 - 1}, \quad z = 1$
6. $\sin \dfrac{1}{z}, \quad z = 0$
7. $\dfrac{\sin \pi z}{4z^2 - 1}, \quad z = \frac{1}{2}$
8. $\dfrac{1 + \cos z}{(z - \pi)^2}, \quad z = \pi$
9. $\dfrac{1}{z^2 - 5z + 6}, \quad z = 2$
10. Show that rule B is correct by applying it to (4.1).
11. Derive (6.2) by using the limit definition of the derivative $h'(z_0)$ instead of using L'Hôpital's rule. Remember that $h(z_0) = 0$ because we are assuming that $f(z)$ has a simple pole at z_0.
12. Prove rule C for finding the residue at a multiple pole, by applying it to (4.1). Note that the rule is valid for $n = 1$ (simple pole) although we seldom use it for that case.

13. Prove rule C by using (3.9). *Hints:* If $f(z)$ has a pole of order n at $z = a$, then $f(z) = g(z)/(z - a)^n$ with $g(z)$ analytic at $z = a$. By (3.9),

$$\int_C \frac{g(z)}{(z - a)}\,dz = 2\pi i g(a)$$

with C a contour inclosing a but no other singularities. Differentiate this equation $(n - 1)$ times with respect to a. (Or, use Problem 3.21.)

Find the residues of the following functions at the indicated points. Try to select the easiest method.

14. $\dfrac{1}{(3z + 2)(2 - z)}$ at $z = -\frac{2}{3}$ and at $z = 2$
15. $\dfrac{1}{(1 - 2z)(5z - 4)}$ at $z = \frac{1}{2}$ and at $z = \frac{4}{5}$
16. $\dfrac{z - 2}{z(1 - z)}$ at $z = 0$ and at $z = 1$
17. $\dfrac{z + 2}{4z^2 - 1}$ at $z = \frac{1}{2}$ and at $z = -\frac{1}{2}$
18. $\dfrac{z + 2}{z^2 + 9}$ at $z = 3i$
19. $\dfrac{\sin^2 z}{2z - \pi}$ at $z = \dfrac{\pi}{2}$
20. $\dfrac{z}{1 - z^4}$ at $z = i$
21. $\dfrac{z^2}{z^4 + 16}$ at $z = \sqrt{2}(1 + i)$
22. $\dfrac{e^{2z}}{1 + e^z}$ at $z = i\pi$
23. $\dfrac{e^{iz}}{9z^2 + 4}$ at $z = \dfrac{2i}{3}$
24. $\dfrac{1 - \cos 2z}{z^3}$ at $z = 0$
25. $\dfrac{e^{2z} - 1}{z^2}$ at $z = 0$
26. $\dfrac{e^{2\pi i z}}{1 - z^3}$ at $z = e^{2\pi i/3}$
27. $\dfrac{\cos z}{1 - 2 \sin z}$ at $z = \pi/6$
28. $\dfrac{z + 2}{(z^2 + 9)(z^2 + 1)}$ at $z = 3i$
29. $\dfrac{e^{2z}}{4 \cosh z - 5}$ at $z = \ln 2$
30. $\dfrac{\cosh z - 1}{z^7}$ at $z = 0$
31. $\dfrac{e^{3z} - 3z - 1}{z^4}$ at $z = 0$
32. $\dfrac{e^{iz}}{(z^2 + 4)^2}$ at $z = 2i$
33. $\dfrac{1 + \cos z}{(\pi - z)^3}$ at $z = \pi$
34. $\dfrac{z - 2}{z^2(1 - 2z)^2}$ at $z = 0$ and at $z = \frac{1}{2}$
35. $\dfrac{z}{(z^2 + 1)^2}$ at $z = i$

14′ to 35′. Use the residue theorem to evaluate the contour integrals of each of the functions in Problems 14 to 35 around a circle of radius $\frac{3}{2}$ and center at the origin. Check carefully to see which singular points are inside the circle. You may use your results in the previous problems as far as they go, but you may have to compute some more residues.

36. For complex z, $J_p(z)$ can be defined by the series (12.9) in Chapter 12. Use this definition to find the Laurent series about $z = 0$ for $z^{-3}J_0(z)$. Find the residue of the function at $z = 0$.

7. EVALUATION OF DEFINITE INTEGRALS BY USE OF THE RESIDUE THEOREM

We are going to use (5.2) and the techniques of Section 6 to evaluate several different types of definite integrals. The methods are best shown by examples.

Example 1. Find $I = \int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}$.

If we make the change of variable $z = e^{i\theta}$, then as θ goes from 0 to 2π, z traverses the unit circle $|z| = 1$ (Figure 7.1) in the counterclockwise direction, and we have a contour integral. We shall evaluate this integral by the residue theorem. If $z = e^{i\theta}$, we have

$$dz = ie^{i\theta}\,d\theta = iz\,d\theta \qquad \text{or} \qquad d\theta = \frac{1}{iz}\,dz,$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2}.$$

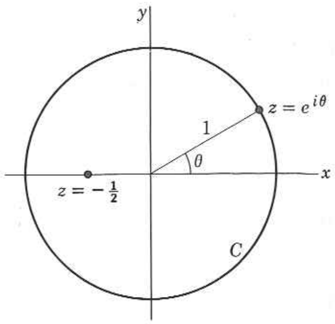


FIGURE 7.1

Making these substitutions in I, we get

$$I = \oint_C \frac{\frac{1}{iz}\,dz}{5 + 2(z + 1/z)} = \frac{1}{i}\oint_C \frac{dz}{5z + 2z^2 + 2}$$

$$= \frac{1}{i}\oint_C \frac{dz}{(2z+1)(z+2)},$$

where C is the unit circle. The integrand has poles at $z = -\frac{1}{2}$ and $z = -2$; only $z = -\frac{1}{2}$ is inside the contour C. The residue of $1/[(2z+1)(z+2)]$ at $z = -\frac{1}{2}$ is

$$R(-\tfrac{1}{2}) = \lim_{z\to -1/2} (z + \tfrac{1}{2}) \cdot \frac{1}{(2z+1)(z+2)} = \left.\frac{1}{2(z+2)}\right|_{z=-1/2} = \tfrac{1}{3}.$$

Then by the residue theorem

$$I = \frac{1}{i}\,2\pi i R(-\tfrac{1}{2}) = 2\pi \cdot \tfrac{1}{3} = \frac{2\pi}{3}.$$

This method can be used to evaluate the integral of any rational function of $\sin\theta$ and $\cos\theta$ between 0 and 2π, provided the denominator is never zero for any value of θ. You can also find an integral from 0 to π if the integrand is even, since the integral from 0 to 2π of an even periodic function is twice the integral from 0 to π of the same function. (See Chapter 7, Section 9 for discussion of even and odd functions.)

Example 2. Evaluate $I = \int_{-\infty}^{\infty} \frac{dx}{1 + x^2}$.

Here we could easily find the indefinite integral and so evaluate I by elementary methods. However, we shall do this simple problem by contour integration to illustrate a method which is useful for more complicated problems.

This time we are not going to make a change of variable in I. We are going to start with a different integral and show how to find I from it. We consider

$$\oint_C \frac{dz}{1 + z^2},$$

where C is the closed boundary of the semicircle shown in Figure 7.2. For any $\rho > 1$, the semicircle incloses the singular point $z = i$ and no others; the residue of the integrand at $z = i$ is

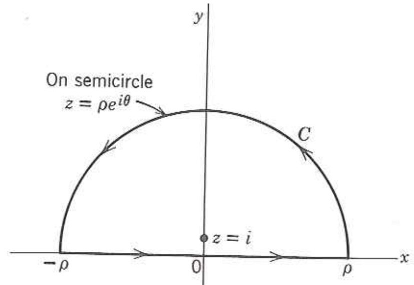


FIGURE 7.2

$$R(i) = \lim_{z\to i}(z - i)\frac{1}{(z-i)(z+i)} = \frac{1}{2i}.$$

Then the value of the contour integral is $2\pi i(1/2i) = \pi$. Let us write the integral in two parts: (1) an integral along the x axis from $-\rho$ to ρ; for this part $z = x$; (2) an integral along the semicircle, where $z = \rho e^{i\theta}$. Then we have

$$\int_C \frac{dz}{1+z^2} = \int_{-\rho}^{\rho} \frac{dx}{1+x^2} + \int_0^{\pi} \frac{\rho i e^{i\theta}\,d\theta}{1 + \rho^2 e^{2i\theta}}. \tag{7.1}$$

We know that the value of the contour integral is π no matter how large ρ becomes since there are no other singular points besides $z = i$ in the upper half-plane. Let $\rho \to \infty$; then the second integral on the right in (7.1) tends to zero since the numerator contains ρ and the denominator ρ^2. Thus the first term on the right tends to π (the value of the contour integral) as $\rho \to \infty$, and we have

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi.$$

This method can be used to evaluate any integral of the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$$

if $P(x)$ and $Q(x)$ are polynomials with the degree of Q at least two greater than the degree of P, and if $Q(z)$ has no real zeros (that is, zeros on the x axis). If the integrand $P(x)/Q(x)$ is an even function, then we can also find the integral from 0 to ∞.

Example 3. Evaluate $I = \int_0^{\infty} \frac{\cos x\,dx}{1 + x^2}$.

We consider the contour integral

$$\oint_C \frac{e^{iz}\,dz}{1 + z^2},$$

where C is the same semicircular contour as in Example 2. The singular point inclosed is again $z = i$, and the residue there is

$$\lim_{z\to i} (z-i)\frac{e^{iz}}{(z-i)(z+i)} = \frac{e^{-1}}{2i} = \frac{1}{2ie}.$$

The value of the contour integral is $2\pi i(1/2ie) = \pi/e$. As in Example 2 we write the contour integral as a sum of two integrals:

$$\oint_C \frac{e^{iz}\,dz}{1+z^2} = \int_{-\rho}^{\rho} \frac{e^{ix}\,dx}{1+x^2} + \int_{\substack{\text{along upper half}\\ \text{of } z=\rho e^{i\theta}}} \frac{e^{iz}\,dz}{1+z^2}. \tag{7.2}$$

As before, we want to show that the second integral on the right of (7.2) tends to zero as $\rho\to\infty$. This integral is the same as the corresponding integral in (7.1) except for the e^{iz} factor. Now

$$|e^{iz}| = |e^{ix-y}| = |e^{ix}|\,|e^{-y}| = e^{-y} \le 1$$

since $y \ge 0$ on the contour we are considering. Since $|e^{iz}| \le 1$, this factor does not change the proof given in Example 2 that the integral along the semicircle tends to zero as the radius $\rho\to\infty$. We have then

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx = \frac{\pi}{e},$$

or taking the real part of both sides of this equation,

$$\int_{-\infty}^{\infty} \frac{\cos x\,dx}{1+x^2} = \frac{\pi}{e}.$$

Since the integrand $(\cos x)/(1+x^2)$ is an even function, the integral from 0 to ∞ is half the integral from $-\infty$ to ∞. Hence we have

$$I = \int_0^{\infty} \frac{\cos x\,dx}{1+x^2} = \frac{\pi}{2e}.$$

Observe that the same proof would work if we replaced e^{iz} by e^{imz} $(m > 0)$ in the above integrals. At the point where we said $e^{-y} \le 1$ (since $y \ge 0$) we would then want $e^{-my} \le 1$ for $y \ge 0$, which is true if $m > 0$. [For $m < 0$, we *could* use a semicircle in the lower half-plane $(y < 0)$; then we would have $e^{my} \le 1$ for $y \le 0$. This is an unnecessary complication, however, in evaluating integrals containing $\sin mx$ or $\cos mx$ since we *can* then choose m to be positive.] Although we have assumed here that (as in Example 2) $Q(x)$ is of degree at least 2 higher than $P(x)$, a more detailed proof (see books on complex variable) shows that degree at least one higher is enough to make the integral

$$\int \frac{P(z)}{Q(z)} e^{imz}\,dz$$

around the semicircle tend to zero as $\rho\to\infty$. Thus

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{imx}\,dx = 2\pi i \cdot \text{sum of the residues of } \frac{P(z)}{Q(z)} e^{imz}$$

in the upper half-plane if all the following requirements are met:

$P(x)$ and $Q(x)$ are polynomials, and
$Q(x)$ has no real zeros, and
the degree of $Q(x)$ is at least 1 greater than the degree of $P(x)$, and
$m > 0$.

By taking real and imaginary parts, we then find the integrals

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos mx\,dx, \qquad \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin mx\,dx.$$

Example 4. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx.$

Here we remove the restriction of Examples 2 and 3 that $Q(x)$ has no real zeros. As in Example 3, we consider

$$\int \frac{e^{iz}}{z}\,dz.$$

To avoid the singular point at $z = 0$, we integrate around the contour shown in Figure 7.3. We then let the radius r shrink to zero so that in effect we are integrating straight through the simple pole at the origin. We are going to show (later in this section and Problem 21) that the net result of integrating in the counterclockwise direction around a closed contour which passes straight* through one or more simple poles is $2\pi i\,\cdot$ (sum of the residues at interior points plus one-half the sum of the residues at the simple poles on the boundary). (*Warning:* this rule does not hold in general for a multiple pole on a boundary.) You might expect this result. If a pole is inside a contour, it contributes $2\pi i\,\cdot$ residue, to the integral; if it is outside, it contributes nothing; if it is *on* the straight line boundary, its contribution is just halfway between zero and $2\pi i\,\cdot$ residue. Using this fact, and observing that, as in Example 3, the integral along the large semicircle tends to zero as R tends to infinity, we have

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = 2\pi i\cdot\tfrac{1}{2}\left(\text{residue of } \frac{e^{iz}}{z} \text{ at } z=0\right) = 2\pi i\cdot\tfrac{1}{2}\cdot 1 = i\pi.$$

Taking the imaginary parts of both sides, we get

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi.$$

* By "straight" we mean that the contour curve has a tangent at the pole, that is, it does not turn a corner there.

To show more carefully that our result is correct, let us return to the contour of Figure 7.3. Since e^{iz}/z is analytic inside this contour, the integral around the whole contour is zero. As we have said, the integral along C tends to zero as $R \to \infty$ by the theorem at the end of Example 3. Along the small semicircle C', we have

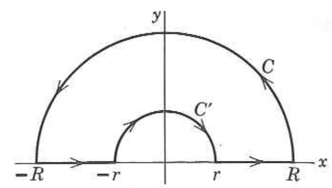


FIGURE 7.3

$$z = re^{i\theta}, \qquad dz = re^{i\theta} i\, d\theta, \qquad \frac{dz}{z} = i\, d\theta,$$

$$\int_{C'} \frac{e^{iz}\, dz}{z} = \int_{C'} e^{iz} i\, d\theta.$$

As $r \to 0$, $z \to 0$, $e^{iz} \to 1$, and the integral (along C' in the direction indicated in Figure 7.3) tends to

$$\int_{\pi}^{0} i\, d\theta = -i\pi.$$

Then we have as $R \to \infty$, and $r \to 0$,

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx - i\pi = 0$$

or

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx = i\pi$$

as before.

Principal Value Taking real and imaginary parts of this equation (and using Euler's formula $e^{ix} = \cos x + i \sin x$), we get

$$\int_{-\infty}^{\infty} \frac{\cos x}{x}\, dx = 0, \qquad \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi.$$

Since $(\sin x)/x$ is an even function, we have

$$\int_{0}^{\infty} \frac{\sin x}{x}\, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

However,

$$\int_{0}^{\infty} \frac{\cos x}{x}\, dx$$

is a divergent integral since the integrand $(\cos x)/x$ is approximately $1/x$ near $x = 0$. The value zero which we found for $I = \int_{-\infty}^{\infty} (\cos x)/x\, dx$ is called the *principal value* (or Cauchy principal value) of I. To see what this means, consider a simpler integral, namely

$$\int_{0}^{5} \frac{dx}{x-3}.$$

The integrand becomes infinite at $x = 3$, and both $\int_0^3 dx/(x-3)$ and $\int_3^5 dx/(x-3)$ are divergent. Suppose we cut out a small symmetric interval about $x = 3$, and integrate from 0 to $3 - r$ and from $3 + r$ to 5. We find

$$\int_{0}^{3-r} \frac{dx}{x-3} = \Big[\ln |x-3| \Big]_{0}^{3-r} = \ln r - \ln 3,$$

$$\int_{3+r}^{5} \frac{dx}{x-3} = \ln 2 - \ln r.$$

The sum of these two integrals is

$$\ln 2 - \ln 3 = \ln \tfrac{2}{3};$$

this sum is independent of r. Thus, if we let $r \to 0$, we get the result $\ln \frac{2}{3}$ which is called the principal value of

$$\int_{0}^{5} \frac{dx}{x-3} \qquad \left(\text{often written } PV \int_{0}^{5} \frac{dx}{x-3} = \ln \tfrac{2}{3}\right).$$

The terms $\ln r$ and $-\ln r$ have been allowed to cancel each other; graphically an infinite area above the x axis and a corresponding infinite area below the x axis have been canceled. In computing the contour integral we integrated along the x axis from $-\infty$ up to $-r$, and from $+r$ to $+\infty$, and then let $r \to 0$; this is just the process we have described for finding principal values, so the result we found for the improper integral $\int_{-\infty}^{\infty} (\cos x)/x\, dx$, namely zero, was the principal value of this integral.

Example 5. Evaluate

$$\int_{0}^{\infty} \frac{r^{p-1}}{1+r}\, dr, \qquad 0 < p < 1,$$

and use the result to prove (5.4) of Chapter 11.

We first find

$$\text{(7.3)} \qquad \oint \frac{z^{p-1}}{1+z}\, dz, \qquad 0 < p < 1, \quad \text{around } C \text{ in Figure 7.4.}$$

Before we can evaluate this integral, we must ask what z^{p-1} means, since for each z there may be more than one value of z^{p-1}. (See discussion of branches at the end of Section 1.) For example, consider the case $p = \frac{1}{2}$; then $z^{p-1} = z^{-1/2}$. Recall from Chapter 2, Section 10, that there are two square roots of any complex number. At a point where $\theta = \pi/4$, say, we have

$$z = re^{i\pi/4}, \qquad z^{-1/2} = r^{-1/2} e^{-i\pi/8}.$$

But if θ increases by 2π (we think of following a circle around the origin and back to our starting point), we have

$$z = re^{i(\pi/4 + 2\pi)}, \qquad z^{-1/2} = r^{-1/2} e^{-i(\pi/8 + \pi)} = -r^{-1/2} e^{-i\pi/8}.$$

Similarly, for any starting point (with $r \neq 0$), we find that $z^{-1/2}$ or z^{p-1} comes back to a different value (different branch) when θ increases by 2π and we return to our starting point. If we want to use the formula z^{p-1} to define a (single-valued) function, we must decide on some interval of length 2π for θ (that is, we must select one branch of z^{p-1}). Let us agree to restrict θ to the values of 0 to 2π in evaluating the contour integral (7.3). We may imagine an artificial barrier or cut (which we agree not to cross) along the positive x axis; this is called a *branch cut*. A point which we cannot encircle (on an arbitrarily small circle) without crossing a branch cut (thus changing to another branch) is called a *branch point*; the origin is a branch point here.

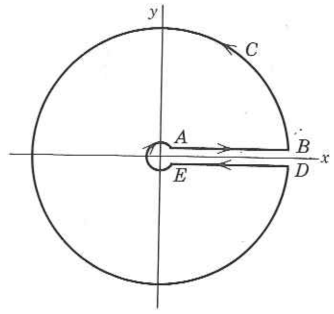


FIGURE 7.4

In Figure 7.4, then, $\theta = 0$ along AB (upper side of the positive x axis); when we follow C around to DE, θ increases by 2π, so $\theta = 2\pi$ on the lower side of the positive x axis. Note that the contour in Figure 7.4 never takes us outside the 0 to 2π interval, so the factor z^{p-1} in (7.3) is a single-valued function. The integrand in (7.3), namely $z^{p-1}/(1+z)$, is now an analytic function inside the closed curve C in Figure 7.4 except for the pole at $z = -1 = e^{i\pi}$. The residue there is $(e^{i\pi})^{p-1} = -e^{i\pi p}$. Then we have

$$(7.4)\qquad \oint_C \frac{z^{p-1}}{1+z}\,dz = -2\pi i e^{i\pi p}, \qquad 0 < p < 1.$$

Along either of the two circles in Figure 7.4 we have $z = re^{i\theta}$ and the integral is

$$\int \frac{r^{p-1}e^{i(p-1)\theta}}{1+re^{i\theta}}\,rie^{i\theta}\,d\theta = i\int \frac{r^p e^{ip\theta}}{1+re^{i\theta}}\,d\theta.$$

This integral tends to zero if $r \to 0$ or if $r \to \infty$. (Verify this; note that the denominator is approximately 1 for small r, and approximately $re^{i\theta}$ for large r.) Thus the integrals along the circular parts of the contour tend to zero as the little circle shrinks to a point and the large circle expands indefinitely. We are left with the two integrals along the positive x axis with AB now extending from 0 to ∞ and DE from ∞ to 0. Along AB we agreed to have $\theta = 0$, so $z = re^{i\cdot 0} = r$, and this integral is

$$\int_{r=0}^{\infty} \frac{r^{p-1}}{1+r}\,dr.$$

Along DE, we have $\theta = 2\pi$, so $z = re^{2\pi i}$ and this integral is

$$\int_{r=\infty}^{0} \frac{(re^{2\pi i})^{p-1}}{1+re^{2\pi i}}\,e^{2\pi i}\,dr = -\int_0^{\infty} \frac{r^{p-1}e^{2\pi ip}}{1+r}\,dr.$$

Adding the AB and DE integrals, we get

$$(1-e^{2\pi ip})\int_0^{\infty}\frac{r^{p-1}}{1+r}\,dr = -2\pi i e^{i\pi p}$$

by (7.4). Then the desired integral is

$$(7.5)\qquad \int_0^{\infty}\frac{r^{p-1}}{1+r}\,dr = \frac{-2\pi i e^{i\pi p}}{1-e^{2\pi ip}} = \frac{\pi\cdot 2i}{e^{i\pi p}-e^{-i\pi p}} = \frac{\pi}{\sin \pi p}.$$

Let us use (7.5) to obtain (5.4) of Chapter 11. Putting $q = 1-p$ in (6.5) and (7.1) of Chapter 11, we have

$$(7.6)\qquad \begin{aligned} B(p, 1-p) &= \int_0^{\infty}\frac{y^{p-1}}{1+y}\,dy \qquad \text{and}\\ B(p, 1-p) &= \Gamma(p)\Gamma(1-p) \qquad \text{since } \Gamma(1) = 1.\end{aligned}$$

Combining (7.5) and (7.6) gives (5.4) of Chapter 11, namely

$$\Gamma(p)\Gamma(1-p) = B(p, 1-p) = \int_0^{\infty}\frac{y^{p-1}}{1+y}\,dy = \frac{\pi}{\sin \pi p}.$$

Argument Principle Since $w = f(z)$ is a complex number for each z, we can write $w = Re^{i\Theta}$ (just as we write $z = re^{i\theta}$) where $R = |w|$ and Θ is the angle of w [or we could call it the angle of $f(z)$]. As z changes, $w = f(z)$ also changes and so R and Θ vary as we go from point to point in the complex (x, y) plane. We want to show that

(a) if $f(z)$ is analytic on and inside a simple closed curve C and $f(z) \neq 0$ on C, then the number of zeros of $f(z)$ inside C is equal to $(1/2\pi)\cdot$ (change in the angle of $f(z)$ as we traverse the curve C);

(b) if $f(z)$ has a finite number of poles inside C, but otherwise meets the requirements stated,* then the change in the angle of $f(z)$ around C is equal to $(2\pi)\cdot$ (the number of zeros minus the number of poles).

(Just as we say that a quadratic equation with equal roots has *two* equal roots, so here we mean that a zero of order n counts as n zeros and a pole of order n counts as n poles.)

To show this we consider

$$\oint_C \frac{f'(z)}{f(z)}\,dz.$$

By the residue theorem, the integral is equal to $2\pi i\cdot$ (sum of the residues at singularities inside C). It is straightforward to show (Problem 42) that the residue of $F(z) = f'(z)/f(z)$ at a zero of $f(z)$ of order n is n, and the residue of $F(z)$ at a pole of $f(z)$ of order p is $-p$. Then if N is the number of zeros and P the number of poles of $f(z)$ inside C, the integral is $2\pi i(N-P)$. Now by direct integration, we have

$$(7.7)\qquad \oint_C \frac{f'(z)}{f(z)}\,dz = \ln f(z)\Big|_C = \ln Re^{i\Theta}\Big|_C = \operatorname{Ln} R\Big|_C + i\Theta\Big|_C,$$

where $R = |f(z)|$ and Θ is the angle of $f(z)$. Recall from Chapter 2, Section 13, that Ln R means the ordinary real logarithm (to the base e) of the positive number R, and

* A function which is analytic except for poles is called *meromorphic*.

is single-valued; $\ln f(z)$ is multiple-valued because Θ is multiple-valued. Then if we integrate from a point A on C all the way around the curve and back to A, $\operatorname{Ln} R$ has the same value at A both at the beginning and at the end, so the term $\operatorname{Ln} R|_C$ is $\operatorname{Ln} R$ at A minus $\operatorname{Ln} R$ at A; this is zero. The same result may not be true for Θ; that is, the angle may have changed as we go from point A all the way around C and back to A. (Think, for example, of the angle of z as we go from $z = 1$ around the unit circle and back to $z = 1$; the angle of z has increased from 0 to 2π.) Collecting our results, we have

(7.8)
$$N - P = \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\, i\Theta_C$$
$$= \frac{1}{2\pi}\cdot(\text{change in the angle of } f(z) \text{ around } C),$$

where N is the number of zeros and P the number of poles of $f(z)$ inside C, with poles of order n counted as n poles and similarly for zeros of order n. Equation (7.8) is known as the *argument principle* (recall from Chapter 2 that *argument* means *angle*).

This principle is often used to find out how many zeros (or poles) a given function has in a given region. (Locating the zeros of a function has important applications to determining the stability of linear systems such as electric circuits and servomechanisms. See, for example, Kuo, page 361, or Kaplan, Operational Methods, Chapter 7.)

Example 6. Let us show that $f(z) = z^3 + 4z + 1 = 0$ at exactly one point in the first quadrant. The closed curve C in (7.8) is, for this problem, the contour OPQ in Figure 7.5, where PQ is a large quarter circle. We first observe that $x^3 + 4x + 1 > 0$ for $x > 0$ and $(iy)^3 + 4iy + 1 \neq 0$ for any y (since its real part, namely 1, $\neq 0$); then $f(z) \neq 0$ on OP or OQ. Also $f(z) \neq 0$ on PQ if we choose a circle large enough to inclose all zeros. We now want to find the change in the angle Θ of $f(z) = Re^{i\Theta}$ as we go around C. Along OP, $z = x$; then $f(z) = f(x)$ is real and so $\Theta = 0$. Along PQ, $z = re^{i\theta}$, with r constant and very large. For very large r, the z^3 term in $f(z)$ far outweighs the other terms, and we have $f(z) \cong z^3 = r^3 e^{3i\theta}$. As θ goes from 0 to $\pi/2$ along PQ, $\Theta = 3\theta$ goes from 0 to $3\pi/2$. On QO, $z = iy$, $f(z) = -iy^3 + 4iy + 1$; then

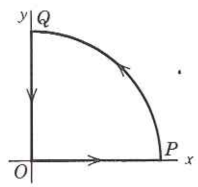


FIGURE 7.5

$$\tan\Theta = \frac{\text{imaginary part of } f(z)}{\text{real part of } f(z)} = \frac{4y - y^3}{1}.$$

For very large y (that is, at Q), we had $\Theta \cong 3\pi/2$ (for $y = \infty$, we would have $\tan\theta = -\infty$, and Θ would be exactly $3\pi/2$). Now as y decreases along QO, the value of $\tan\Theta = 4y - y^3$ decreases in magnitude but remains negative until it becomes 0 at $y = 2$. This means that Θ changes from $3\pi/2$ to 2π. Between $y = 2$ and $y = 0$, the tangent becomes positive, but then decreases to zero again without becoming infinite. This means that the angle Θ increases beyond 2π but not as far as $2\pi + \pi/2$, and then

decreases again to 2π. Thus the total change in Θ around C is 2π, and by (7.8), the number of zeros of $f(z)$ in the first quadrant is $(1/2\pi)\cdot 2\pi = 1$. If we realize that (for a polynomial with real coefficients) the zeros off the real axis always occur in conjugate pairs, we see that there must also be one zero for z in the fourth quadrant, and the third zero must be on the negative x axis.

PROBLEMS, SECTION 7

Using one of the methods discussed in Examples 1, 2, and 3, evaluate the following definite integrals.

1. $\displaystyle\int_0^{2\pi} \frac{d\theta}{13 + 5\sin\theta}$

2. $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta}$

3. $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 - 4\sin\theta}$

4. $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta\,d\theta}{5 + 3\cos\theta}$

5. $\displaystyle\int_0^{\pi} \frac{d\theta}{1 - 2r\cos\theta + r^2}$ $\quad (0 \le r < 1)$

6. $\displaystyle\int_0^{\pi} \frac{d\theta}{(2 + \cos\theta)^2}$

7. $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta\,d\theta}{5 + 4\cos\theta}$

8. $\displaystyle\int_0^{\pi} \frac{\sin^2\theta\,d\theta}{13 - 12\cos\theta}$

9. $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 + \sin\theta\cos\alpha}$ $\quad (\alpha = \text{const.})$

10. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2 + 4x + 5}$

11. $\displaystyle\int_0^{\infty} \frac{dx}{(4x^2 + 1)^3}$

12. $\displaystyle\int_0^{\infty} \frac{x^2\,dx}{x^4 + 16}$

13. $\displaystyle\int_0^{\infty} \frac{x^2\,dx}{(x^2 + 4)(x^2 + 9)}$

14. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x^2 + 4x + 5}$

15. $\displaystyle\int_0^{\infty} \frac{\cos 2x\,dx}{9x^2 + 4}$

16. $\displaystyle\int_0^{\infty} \frac{x\sin x\,dx}{9x^2 + 4}$

17. $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x\,dx}{x^2 + 4x + 5}$

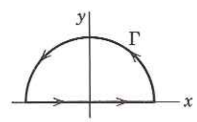
18. $\displaystyle\int_0^{\infty} \frac{\cos\pi x\,dx}{1 + x^2 + x^4}$

19. $\displaystyle\int_0^{\infty} \frac{\cos 2x\,dx}{(4x^2 + 9)^2}$

20. $\displaystyle\int_0^{\infty} \frac{\cos x\,dx}{(1 + 9x^2)^2}$

21. In Example 4 we stated a rule for evaluating a contour integral when the contour passes through simple poles. We proved that the result was correct for

$$\int_\Gamma \frac{e^{iz}}{z}\,dz$$



around the contour Γ shown here.

(a) By following the same method (integrating around C' of Figure 7.3 and letting $r \to 0$) show that the result is correct if we replace e^{iz} by any $f(z)$ which is analytic at $z = 0$.

(b) Repeat the proof in (a) for

$$\int_\Gamma \frac{f(z)}{(z-a)}\,dz, \qquad a \text{ real}$$

(that is, a pole on the x axis), with $f(z)$ analytic at $z = a$.

Using the rule of Example 4 (also see Problem 21), evaluate the following integrals. Find principal values if necessary.

22. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x-1)(x^2+1)}$

23. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)(2-x)}$

24. $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin \pi x}{1-x^2}\,dx$

25. $\displaystyle\int_{0}^{\infty} \frac{x \sin x}{9x^2-\pi^2}\,dx$

26. $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{(x-1)^4-1}$

27. $\displaystyle\int_{0}^{\infty} \frac{\cos \pi x}{1-4x^2}\,dx$

28. $\displaystyle\int_{0}^{\infty} \frac{dx}{1-x^4}$

29. $\displaystyle\int_{0}^{\infty} \frac{\sin ax}{x}\,dx$

30. (a) By the method of Example 2 evaluate $\displaystyle\int_0^\infty \frac{dx}{1+x^4}$.

(b) Evaluate the same integral by using tables to get the indefinite integral; unless you are very careful you may get zero. Explain why.

(c) Make the change of variables $u = x^4$ in the integral in (a) and evaluate the u integral using (7.5).

31. Use the method of Problem 30(c) to evaluate $\displaystyle\int_0^\infty \frac{dx}{1+x^6}$.

32. Use the method of Problem 30(c) and the contour and method of Example 5, to evaluate

$$\int_0^\infty \frac{dx}{(1+x^4)^2}.$$

Evaluate the following integrals by the method of Example 5.

33. $\displaystyle\int_0^\infty \frac{\sqrt{x}\,dx}{1+x^2}$

34. $\displaystyle\int_0^\infty \frac{\sqrt{x}\,dx}{(1+x)^2}$

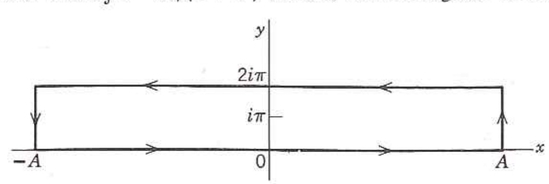
35. $\displaystyle\int_0^\infty \frac{x^{1/3}\,dx}{(1+x)(2+x)}$

36. $\displaystyle\int_0^\infty \frac{\ln x}{x^{3/4}(1+x)}\,dx$

37. (a) Show that

$$\int_{-\infty}^{\infty} \frac{e^{px}}{1+e^x}\,dx = \frac{\pi}{\sin \pi p}$$

for $0 < p < 1$. *Hint:* Find $\int e^{pz}\,dz/(1+e^z)$ around the rectangular contour shown.



Show that the integrals along the vertical sides tend to zero as $A \to \infty$. Note that the integral along the upper side is a multiple of the integral along the x axis.

(b) Make the change of variable $y = e^x$ in the x integral of part (a), and using (6.5) of Chapter 11, show that this integral is the beta function, $B(p, 1-p)$. Then using (7.1) of Chapter 11, show that $\Gamma(p)\Gamma(1-p) = \pi/\sin \pi p$.

38. Using the same contour and method as in Problem 37a evaluate

$$\int_{-\infty}^{\infty} \frac{e^{px}}{1-e^x}\,dx, \qquad 0 < p < 1.$$

Hint: The only difference between this problem and Problem 37a is that you now have two simple poles on the contour instead of a pole inside. Use the rule of Example 4.

39. Evaluate

$$\int_{-\infty}^{\infty} \frac{e^{2\pi x/3}}{\cosh \pi x}\,dx.$$

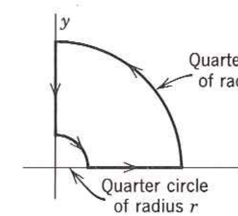
Hint: Use a rectangle as in Problem 37a but of height 1 instead of 2π. Note that there is a pole at $i/2$.

40. Evaluate

$$\int_0^\infty \frac{x\,dx}{\sinh x}.$$

Hint: First find the $-\infty$ to ∞ integral. Use a rectangle of height π and note the simple pole at $i\pi$ on the contour.

41. The Fresnel integrals, $\int_0^u \sin u^2\,du$ and $\int_0^u \cos u^2\,du$, are important in optics. For the case of infinite upper limits, evaluate these integrals as follows: Make the change of variable $x = u^2$; to evaluate the resulting integrals, find $\oint z^{-1/2} e^{iz}\,dz$ around the contour shown. Let $r \to 0$ and $R \to \infty$ and show that the integrals along these quarter-circles tend to zero. Recognize the integral along the y axis as a Γ function and so evaluate it. Hence evaluate the integral along the x axis; the real and imaginary parts of this integral are the integrals you are trying to find.



42. If $F(z) = f'(z)/f(z)$,

(a) show that the residue of $F(z)$ at an nth order zero of $f(z)$, is n. *Hint:* If $f(z)$ has a zero of order n at $z = a$, then

$$f(z) = a_n(z-a)^n + a_{n+1}(z-a)^{n+1} + \cdots.$$

(b) Also show that the residue of $F(z)$ at a pole of order p of $f(z)$, is $-p$. *Hint:* See the definition of a pole of order p at the end of Section 4.

43. By using theorem (7.8), show that $z^3 + z^2 + 9 = 0$ has exactly one root in the first quadrant. Recall that the roots of a polynomial equation with real coefficients are either real or occur in conjugate pairs $a \pm bi$ (think of the quadratic formula, for example). Hence show that since $z^3 + z^2 + 9 = 0$ has one root in the first quadrant, it has one in the fourth and one on the negative real axis.

44. The *fundamental theorem of algebra* says that every equation of the form $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = 0$, $a_n \neq 0$, $n \geq 1$, has at least one root, from which it follows that an nth degree equation has n roots. Prove this by using the argument principle. *Hint:* Follow the increase in the angle of $f(z)$ around a very large circle $z = re^{i\theta}$; for sufficiently large r, all roots are inclosed, and $f(z)$ is approximately $a_n z^n$.

As in Problem 43 find out in which quadrants the roots of the following equations lie:

45. $z^3 + z^2 + z + 4 = 0$

46. $z^3 + 3z^2 + 4z + 2 = 0$

47. $z^3 + 4z^2 + 12 = 0$

48. $z^4 - z^3 + 6z^2 - 3z + 5 = 0$

49. $z^4 - 4z^3 + 11z^2 - 14z + 10 = 0$

50. $z^4 + z^3 + 4z^2 + 2z + 3 = 0$

51. Use (7.8) to evaluate

$$\oint_C \frac{f'(z)}{f(z)}\, dz, \qquad \text{where } f(z) = \frac{z^3(z+1)^2 \sin z}{(z^2+1)^2(z-3)},$$

around the circle $|z| = 2$; around $|z| = \frac{1}{2}$.

52. Use (7.8) to evaluate $\oint \frac{z^3\, dz}{1 + 2z^4}$ around $|z| = 1$.

53. Use (7.8) to evaluate $\oint \frac{z^3 + 4z}{z^4 + 8z^2 + 16}\, dz$ around the circle $|z - 2i| = 2$.

54. Use (7.8) to evaluate

$$\oint_C \frac{\sec^2(z/4)\, dz}{1 - \tan(z/4)},$$

where C is the rectangle formed by the lines $y = \pm 1$, $x = \pm \frac{5}{2}\pi$.

8. THE POINT AT INFINITY; RESIDUES AT INFINITY

It is often useful to think of the complex plane as corresponding to the surface of a sphere in the following way. In Figure 8.1, the sphere is tangent to the plane at the origin O. Let O be the south pole of the sphere, and N be the north pole of the sphere. If a line through N intersects the sphere at P and the plane at Q, we say that the point P on the sphere and the point Q on the plane are corresponding points. Then we have a one-to-one correspondence between points on the sphere (except N) and points of the plane (at finite distances from O). Imagine point Q moving farther and farther out away from O; then P moves nearer and nearer to N. If $z = x + iy$ is the complex coordinate of Q, then as Q moves out farther and farther from O, we would say $z \to \infty$. It is customary to say that the point N corresponds to the *point at infinity* in the complex plane. Observe that straight lines through the origin in the plane correspond to meridians of the sphere. The meridians all pass through both the north pole and

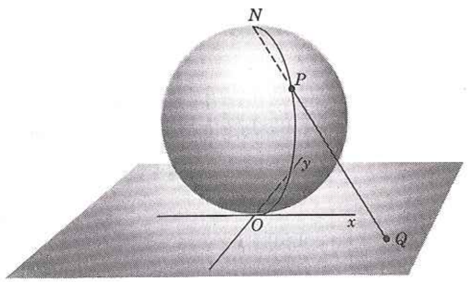


FIGURE 8.1

the south pole. Corresponding to this, straight lines through the origin in the complex plane pass through the point at infinity. Circles in the complex plane with center at O correspond to parallels of latitude on the sphere. This mapping of the complex plane onto a sphere (or the mapping of the sphere onto a tangent plane) is called a *stereographic projection.*

To investigate the behavior of a function at infinity, we replace z by $1/z$ and consider how the new function behaves at the origin. We then say that infinity is a regular point, a pole, etc., of the original function, depending on what the new function does at the origin. For example, consider z^2 at infinity; $1/z^2$ has a pole of order 2 at the origin, so z^2 has a pole of order 2 at infinity. Or consider $e^{1/z}$; since e^z is analytic at $z = 0$, $e^{1/z}$ is analytic at ∞.

Next we want to see how to find the residue of a function at ∞. To do this, we are going to want to replace z by $1/z$ and work around the origin. In order to keep our notation straight, let us use two variables, namely Z which takes on values near ∞, and $z = 1/Z$ which takes on values near 0. The residue of a function at ∞ is defined so that the residue theorem holds, that is,

$$\oint_C f(Z)\, dZ = 2\pi i \cdot (\text{residue of } f(Z) \text{ at } Z = \infty) \tag{8.1}$$

if C is a closed path around the point at ∞ but inclosing no other singular points. Now what does it mean to integrate "around ∞"? Recall that we have agreed to traverse contours so that the area inclosed always lies to our left. The area we wish to "inclose" is the area "around ∞"; if C is a circle, this area would lie *outside* the circle in our usual terminology. Figure 8.1 may clarify this. Imagine a small circle about the north pole; the area inside this circle (that is, the area including N) corresponds to points in the plane which are outside a large circle C. We must go around C in the clockwise direction in order to have the area "around ∞" to our left. This is indicated by the arrow on the integral sign in (8.1). Note that if $Z = Re^{i\Theta}$, then in going clockwise around C, we are going in the direction of *decreasing* Θ. Let us make the following change of variable in the integral (8.1):

$$Z = \frac{1}{z}, \qquad dZ = -\frac{1}{z^2}\, dz.$$

If $Z = Re^{i\Theta}$ traverses a circle C of radius R in the direction of decreasing Θ, then $z = 1/Z = (1/R)e^{-i\Theta} = re^{i\theta}$ traverses a circle C' of radius $r = 1/R$ in the counterclockwise direction (that is, $\theta = -\Theta$ increases as Θ decreases). Thus (8.1) becomes

$$\oint_{C'} -\frac{1}{z^2} f\left(\frac{1}{z}\right) dz = 2\pi i \cdot (\text{residue of } f(Z) \text{ at } Z = \infty). \tag{8.2}$$

The integral in (8.2) is an integral about the origin and so can be evaluated by calculating the residue of $(-1/z^2)f(1/z)$ at the origin. (There are no other singular points of $f(1/z)$ inside C' because we assumed that there were no singular points of $f(Z)$ outside C except perhaps ∞.) Thus we have

$$(\text{residue of } f(Z) \text{ at } Z = \infty) = -\left(\text{residue of } \frac{1}{z^2} f\left(\frac{1}{z}\right) \text{ at } z = 0\right) \tag{8.3}$$

and we can use the methods we already know for computing residues at the origin.