Math 303: Quiz # 4

Fall 2017

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 45 minutes.
- **Give details of your response to each problem. You will not be given any credit, if it is not clear how you have obtained your response.**

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1 (10 points) Express $|x-\pi|$ in terms of the Heaviside step function: $\theta(x) := \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$, and use the result to compute $\frac{d^2}{dx^2}|x-\pi|$ for all $x \in \mathbb{R}$.

$$|x-\pi| = (x-\pi)\theta(x-\pi) + (-x+\pi)\theta(\pi-x)$$

$$\frac{d}{dx}|x-\pi| = \theta(x-\pi) + \underbrace{(x-\pi)\delta(x-\pi)}_{0} - \theta(\pi-x) - \underbrace{(-x+\pi)\delta(\pi-x)}_{0}$$

$$= \theta(x-\pi) - \theta(\pi-x)$$

$$\frac{d^2}{dx^2}|x-\pi| = \delta(x-\pi) - [-\delta(\pi-x)]$$

$$= \delta(x-\pi) + \underbrace{\delta(\pi-x)}_{=\,\delta(x-\pi)}$$

$$= 2\delta(x-\pi)$$

2 (10 points) Let $\theta(x)$ be the Heaviside step function. Calculate $\frac{d}{dx}\left[\theta(\pi-x)\sin x+\theta(x-\pi)\cos x\right]$.

$$= -\delta(\pi-x)\sin x + \theta(\pi-x)\cos x + \delta(x-\pi)\cos x - \theta(x-\pi)\sin x$$

$$= -\delta(\pi-x)\underbrace{\sin\pi}_{0} + \theta(\pi-x)\cos x + \delta(x-\pi)\underbrace{\cos(\pi)}_{-1} - \theta(x-\pi)\sin x$$

$$= \theta(\pi-x)\cos x - \theta(x-\pi)\sin x - \delta(x-\pi)$$

3 (10 points) Let $\delta(x)$ be the Dirac delta function. Calculate $\int_{-1}^{5}\cos(\frac{\pi x}{4})\,\delta(x^3+3x^2+2x)dx$

$$f(x) = x(x^2+3x+2) = x(x+1)(x+2)$$

$$f'(x) = (x+1)(x+2) + x(x+2) + x(x+1)$$

$$f(x)=0 \Rightarrow x=0,\quad x=-1,\quad x=-2$$

$$f'(0)=2,\quad f'(-1)=-1,\quad f'(-2)=2$$

$$\delta(f(x)) = \frac{\delta(x)}{|f'(0)|} + \frac{\delta(x+1)}{|f'(-1)|} + \frac{\delta(x+2)}{|f'(-2)|}$$

$$= \frac{1}{2}\delta(x) + \delta(x+1) + \frac{1}{2}\delta(x+2)$$

$$\Rightarrow \int_{-1}^{5}\cos(\frac{\pi x}{4})\,\delta(x^3+3x^2+2x)dx = \int_{-1}^{5}\cos(\frac{\pi x}{4})\left[\frac{1}{2}\delta(x)+\delta(x+1)+\frac{1}{2}\delta(x+2)\right]dx$$

$$-2 \notin [-1,5]$$

$$= \frac{1}{2}\cos(0) + \cos(-\frac{\pi}{4})$$

$$= \frac{1}{2} + \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2}+1}{2}$$

4 (10 points) Compute the Fourier transform of $f(x) := \sqrt{2\pi}\, x + e^x\theta(1-x^2)$, where $\theta(x)$ is the Heaviside step function.

$$\mathcal{F}\{f(x)\} = \sqrt{2\pi}\,\mathcal{F}\{x\} + \mathcal{F}\{e^x\theta(1-x^2)\}$$

$$\mathcal{F}\{x\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\, x\, dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} i\left[\frac{\partial}{\partial k} e^{-ikx}\right] dx$$

$$= \frac{i}{\sqrt{2\pi}}\frac{d}{dk}\underbrace{\int_{-\infty}^{\infty} e^{-ikx}\, dx}_{2\pi\delta(k)}$$

$$= \sqrt{2\pi}\, i\, \delta'(k)$$

$$\mathcal{F}\{e^x\theta(1-x^2)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} e^x\theta(1-x^2)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{(-ik+1)x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left.\frac{e^{(-ik+1)x}}{-ik+1}\right|_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{-ik+1}}{-ik+1} - \frac{e^{ik-1}}{-ik+1}\right)$$

$$= \frac{e^{-ik+1} - e^{ik-1}}{\sqrt{2\pi}(1-ik)}$$

$$\Rightarrow \boxed{\mathcal{F}\{f(x)\} = 2\pi i\,\delta'(k) + \frac{e^{1-ik} - e^{ik-1}}{\sqrt{2\pi}(1-ik)}}$$