Math 303: Quiz # 2

Fall 2017

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 45 minutes.
- **Give details of your response to each problem. You will not be given any credit, if it is not clear how you have obtained your response.**

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1 (5 points) Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be functions with second partial derivatives. Use the properties of the Levi Civita symbol ϵ_{ijk} to show that $\vec{\nabla} \cdot (\vec{\nabla} f \times \vec{\nabla} g) = 0$.

Warning: To get full credit you must explain in words the properties of ϵ_{ijk} that you use in each step of your proof.

$$\vec{\nabla} \cdot (\vec{\nabla} f \times \vec{\nabla} g) = \sum_{i=1}^{3} \partial_i \sum_{j,k=1}^{3} \epsilon_{ijk}\, \partial_j f\, \partial_k g$$

$$= \sum_{i,j,k=1}^{3} \epsilon_{ijk}\, \partial_i (\partial_j f\, \partial_k g)$$

$$= \sum_{i,j,k=1}^{3} \epsilon_{ijk} \left[(\partial_i \partial_j f)(\partial_k g) + (\partial_j f)(\partial_i \partial_k g) \right]$$

$$= \sum_{k=1}^{3} \partial_k g \underbrace{\sum_{i,j=1}^{3} \underbrace{\epsilon_{ijk}}_{\text{antisymmetric}} \underbrace{\partial_i \partial_j f}_{\text{symmetric}}}_{0} + \sum_{j=1}^{3} \partial_j f \underbrace{\sum_{i,k=1}^{3} \underbrace{\epsilon_{ijk}}_{\text{antisym}} \underbrace{\partial_i \partial_k g}_{\text{Sym.}}}_{0}$$

$$= 0$$

2 (20 points) Find the largest and smallest values of the function:

$$f(x,y,z)=e^{-z^2}(x^2-2y^2+x+1)$$

in the region of space given by $V := \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2\le 1\}$.

First explore inside of V $(x^2+y^2<1)$:

$0=\frac{\partial f}{\partial x}=e^{-z^2}(2x+1) \Rightarrow \boxed{x=-\frac{1}{2}}$ $\Rightarrow x^2-2y^2+x+1=\frac{1}{4}-\frac{1}{2}+1=\frac{3}{4}$

$0=\frac{\partial f}{\partial y}=e^{-z^2}(-4y) \Rightarrow \boxed{y=0}$

$0=\frac{\partial f}{\partial z}=-2ze^{-z^2}(x^2-2y^2+x+1) \longrightarrow \boxed{z=0}$

So we find the point $\boxed{\vec{P}_0=(-\frac{1}{2},0,0)\in V}$ and $\boxed{f(\vec{P}_0)=\frac{3}{4}}$

Now we explore ∂V $(x^2+y^2=1)$: We use Lagrange's multiplier method

let $F(x,y,z)=f(x,y,z)+\lambda(x^2+y^2-1)$

$0=\frac{\partial f}{\partial x}=e^{-z^2}(2x+1)+2\lambda x \Rightarrow x\neq 0$ and $2\lambda=-\frac{e^{-z^2}(2x+1)}{x}$ ①

$0=\frac{\partial f}{\partial y}=e^{-z^2}(-4y)+2\lambda y \Rightarrow \begin{cases}\boxed{y=0} & ② \\ \text{or} \\ 2\lambda=-\frac{e^{-z^2}(-4y)}{y}=4e^{-z^2} & ③\end{cases}$

$0=\frac{\partial f}{\partial z}=-2ze^{-z^2}(x^2-2y^2+x+1)$ ④

$0=\frac{\partial f}{\partial \lambda}=x^2+y^2-1$ ⑤

② & ⑤ $\Rightarrow \boxed{x=\pm 1}$ ⑥ & ④ $\Rightarrow -2ze^{-z^2}(1\pm 1+1)=0 \Rightarrow \boxed{z=0}$ ⑦

② & ⑥ & ⑦ $\Rightarrow$ we find two points $\boxed{\vec{P}_{1,2}=(\pm 1,0,0)\in\partial V}$

and $\boxed{f(\vec{P}_1)=3}$ and $\boxed{f(\vec{P}_2)=1}$

① & ③ $\Rightarrow -\frac{2x+1}{x}=4 \Rightarrow 2x-1=4x \Rightarrow \boxed{x=-\frac{1}{6}}$ ⑧

⑧ & ⑤ $\Rightarrow \boxed{y=\pm\sqrt{1-\frac{1}{36}}=\pm\frac{\sqrt{35}}{6}}$ ⑨

8 & 9 $\Rightarrow x^2 - 2y^2 + x + 1 = \frac{1}{36} - \frac{70}{36} - \frac{1}{6} + 1 = \frac{-39}{36} = -\frac{13}{12}$ (10)

(4) & (10) $\Rightarrow \boxed{z = 0}$ (11)

8 & 9 & 11 $\Rightarrow$ we find another points $\boxed{\vec{P}_{3,4} = \left(-\frac{1}{6}, \pm\frac{\sqrt{35}}{6}, 0\right) \in \partial V}$

and $\boxed{f(\vec{P}_{3,4}) = -\frac{39}{36}}$

As $z \longrightarrow \pm\infty$, $f(x,y,z) \longrightarrow 0$

So the Largest value of f is 3 which is attained at $\vec{P}_1 = (+1, 0, 0)$

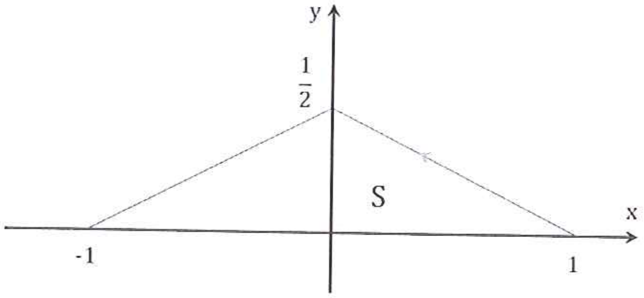
and the smallest value of f is $-\frac{13}{12}$ which is attained at

the points $\vec{P}_{3,4}$ on ∂V.

3 (15 points) Let $\mathbf{i}$ and $\mathbf{j}$ be the unit vectors pointing along the positive x- and y-axes in $\mathbb{R}^2$, $\mathbf{J} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\mathbf{J}(x, y) = (4x^3y^3 + \sin y - x)\mathbf{i} + (-3x^2y^4 - \cos x + 3)\mathbf{j},$$

and S be the region in $\mathbb{R}^2$ that is bounded by a triangle with vertices $(1, 0)$, $(0, \frac{1}{2})$ and $(-1, 0)$, as shown in the figure.



Use Divergence theorem in two dimensions to evaluate $\oint_{\partial S} \mathbf{J} \cdot \hat{\mathbf{n}}\, d\ell$, where ∂S is the counterclockwise-oriented boundary of S and $\hat{\mathbf{n}}$ is the outward unit normal vector to ∂S.

$$I := \oint_{\partial S} \vec{J} \cdot \hat{n}\, d\ell = \iint_S \vec{\nabla} \cdot \vec{J}\, d\sigma \qquad \vec{\nabla} \cdot \vec{J} = \frac{\partial}{\partial x} J_1 + \frac{\partial}{\partial y} J_2$$

$$\frac{\partial J_1}{\partial x} = 12x^2y^3 - 1, \qquad \frac{\partial J_2}{\partial y} = -12x^2y^3$$

$$\Rightarrow \quad \vec{\nabla} \cdot \vec{J} = -1$$

$$\Rightarrow \quad I = -\underbrace{\iint_S d\sigma}_{\text{Area of } S \,=\, \frac{1}{2}\left(\frac{1}{2} \times 2\right) = \frac{1}{2}} = -\frac{1}{2}$$