Solutions

Math 303: Quiz # 1

Fall 2017

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 45 minutes.
- **Give details of your response to each problem. You will not be given any credit, if it is not clear how you have obtained your response.**

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1 (10 points) Find real numbers m, n, and α such that $\Omega = xy^m dx + \alpha x^n y^2 dy$ is an exact differential, i.e., there is a function f such that $\Omega = df$.

$$\Omega = \underbrace{xy^m}_{\Omega_1} dx + \underbrace{\alpha x^n y^2}_{\Omega_2} dy$$

$$\Omega = df \iff \frac{\partial}{\partial y}\Omega_1 = \frac{\partial}{\partial x}\Omega_2$$

$$\iff m x y^{m-1} = n\alpha x^{n-1} y^2$$

$$\iff \begin{array}{lll} n-1=1 & \Rightarrow & n=2 \\ m-1=2 & \Rightarrow & m=3 \\ m=n\alpha & \Rightarrow & \alpha = \frac{m}{n} = \frac{3}{2} \end{array}$$

2 (20 points) Find the real and imaginary parts of all possible values of $\sin(\ln(1+i))$. To get full credit you must simplify your response as much as possible.

$$\sin[\ln(1+i)] = \frac{e^{i\ln(1+i)} - e^{-i\ln(1+i)}}{2i}$$

$$1+i = \sqrt{2}\, e^{\frac{i\pi}{4} + 2\pi n i} = e^{\ln\sqrt{2} + i(\frac{\pi}{4} + 2\pi n)}$$

$$\Rightarrow \ln(1+i) = \ln\sqrt{2} + i(\tfrac{\pi}{4} + 2\pi n) \qquad n \in \mathbb{Z}$$

$$\Rightarrow e^{\pm i\ln(1+i)} = e^{\pm i\ln\sqrt{2} \mp (\frac{\pi}{4} + 2\pi n)}$$

$$= e^{\mp(\frac{\pi}{4} + 2\pi n)}\left[\cos\ln\sqrt{2} \pm i\sin\ln\sqrt{2}\right]$$

$$\Rightarrow \sin[\ln(1+i)] = \frac{1}{2i}\left\{ e^{-(\frac{\pi}{4} + 2\pi n)}\left[\cos\ln\sqrt{2} + i\sin\ln\sqrt{2}\right] - e^{\frac{\pi}{4} + 2\pi n}\left[\cos\ln\sqrt{2} - i\sin\ln\sqrt{2}\right]\right\}$$

$$= -\frac{i}{2}\cos(\ln\sqrt{2})\left(e^{-(\frac{\pi}{4} + 2\pi n)} - e^{\frac{\pi}{4} + 2\pi n}\right) + \frac{1}{2}\sin(\ln\sqrt{2})\left(e^{-(\frac{\pi}{4} + 2\pi n)} + e^{\frac{\pi}{4} + 2\pi n}\right)$$

$$= \sin(\ln\sqrt{2})\cosh(\tfrac{\pi}{4} + 2\pi n) + i\cos(\ln\sqrt{2})\sinh(\tfrac{\pi}{4} + 2\pi n)$$

So $\mathrm{Re}\{\sin[\ln(1+i)]\} = \sinh(\ln\sqrt{2})\cosh(\frac{\pi}{4} + 2\pi n)$

& $\mathrm{Im}\{\sin[\ln(1+i)]\} = \cosh(\ln\sqrt{2})\sinh(\frac{\pi}{4} + 2\pi n)$

Note also that $\ln\sqrt{2} = \frac{1}{2}\ln 2$

3 (10 points) Calculate the Jacobian $\dfrac{\partial(y_1, y_2)}{\partial(x_1, x_2)}$ of the coordinate transformation: $(x_1, x_2) \to (y_1, y_2)$, where

$$y_1 := x_1 \cos(x_2^2) + x_2 \sin(x_2^2), \qquad y_2 := -x_1 \sin(x_2^2) + x_2 \cos(x_2^2).$$

Recall that $\dfrac{\partial(y_1, y_2)}{\partial(x_1, x_2)}$ is the determinant of the matrix formed by the partial derivatives $\dfrac{\partial y_i}{\partial x_j}$.

$$J = \frac{\partial(Y_1, Y_2)}{\partial(X_1, X_2)} = \begin{vmatrix} \dfrac{\partial Y_1}{\partial X_1} & \dfrac{\partial Y_2}{\partial X_1} \\ \dfrac{\partial Y_1}{\partial X_2} & \dfrac{\partial Y_2}{\partial X_2} \end{vmatrix}$$

$$\frac{\partial Y_1}{\partial X_1} = \cos(X_2^2)$$

$$\frac{\partial Y_1}{\partial X_2} = -2X_1X_2 \sin(X_2^2) + \sin(X_2^2) + 2X_2^2 \cos(X_2^2)$$

$$\frac{\partial Y_2}{\partial X_1} = -\sin(X_2^2)$$

$$\frac{\partial Y_2}{\partial X_2} = -2X_1X_2 \cos(X_2^2) + \cos(X_2^2) - 2X_2^2 \sin(X_1^2)$$

$$\Rightarrow J = -2X_1X_2 \cos^2(X_2^2) + \cos^2(X_2^2) - 2X_2^2 \sin(X_2^2)\cos(X_1^2) -$$
$$-\left[2X_1X_2 \sin^2(X_2^2) - \sin^2(X_2^2) - 2X_2^2 \sin(X_2^2)\cos(X_2^2)\right]$$

$$= -2X_1X_2 + 1$$