Math 303: Midterm Exam

November 18, 2017

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 2.5 hours.
- You must show the details of all your work. Illegible and ambiguous explanations and calculations will lead to deductions from your grade.
- You may use the option of grading your own work. If your estimated grade differs from your actual grade by less than 10 points, you will be given the higher of the two.

Estimated Grade:	
Actual Grade:	
Adjusted Grade:	

Problem 1 (10 points) Let $\phi : \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ be functions with first partial derivatives. Use the properties of the Levi Civita symbol ϵ_{ijk} to show that

$$\nabla \times (\phi\ \mathbf{A}) = \phi\, \nabla \times \mathbf{A} - \mathbf{A} \times (\nabla \phi).$$

$$(\vec{\nabla} \times (\phi \vec{A}))_i = \sum_{j,k=1}^{3} \epsilon_{ijk}\, \partial_j (\phi A_k)$$

$$= \sum_{j,k=1}^{3} \epsilon_{ijk} [(\partial_j \phi) A_k + \phi\, \partial_j A_k]$$

$$= \underbrace{\sum_{j,k=1}^{3} \phi\, \epsilon_{ijk}\, \partial_j A_k}_{\phi (\vec{\nabla} \times \vec{A})_i} + \underbrace{\sum_{j,k=1}^{3} \underbrace{\epsilon_{ijk}}_{-\epsilon_{ikj}} (\partial_j \phi) A_k}_{\underbrace{-\sum_{k,j=1}^{3} \epsilon_{ijk}\, \partial_k \phi\, A_j}_{\underbrace{-\sum_{j,k=1}^{3} \epsilon_{ijk}\, A_j\, \partial_k \phi}_{-(A \times \vec{\nabla}\phi)_i}}}$$

$$= [\phi (\vec{\nabla} \times \vec{A}) - (A \times \vec{\nabla} \phi)]_i$$

$$\Rightarrow \quad \vec{\nabla} \times (\phi \vec{A}) = \phi (\vec{\nabla} \times \vec{A}) - A \times \vec{\nabla} \phi .$$

Problem 2 Let $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $\mathbf{A}(x, y, z) := [x^2y + z\cos(xz)]\mathbf{i} + x\,\mathbf{j} + [z + x\cos(xz)]\mathbf{k}$ and

$$S := \{(x, y, z) \in \mathbb{R}^3 \mid z + x^2 + y^2 = 4 \text{ and } z \geq 0\}.$$

2.a (5 points) Compute the curl of $\mathbf{A}$, i.e., $\nabla \times \mathbf{A}$.

$$\nabla \times \bar{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x^2y + z\cos(xz) & x & z + x\cos(xz) \end{vmatrix}$$

$$= \hat{i}\,(0 - 0) + (-\hat{j})\,[\cos(xz) - x\sin(xz) - \cos(xz) + xz\sin(xz)]$$

$$+ \hat{k}\,[1 - x^2]$$

$$= (1 - x^2)\,\hat{k}$$

2.b (10 points) Evaluate the surface integral $I := \int\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, d\sigma$ without using the Stokes theorem. Here $\mathbf{n}$ is the unit normal outward vector to S and $d\sigma$ is the surface element of S.

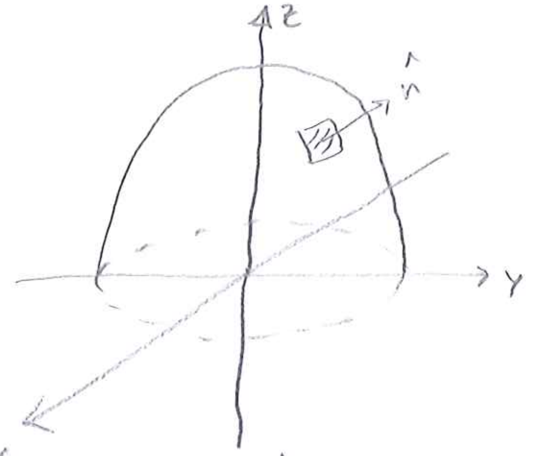
$$z = -(x^2+y^2)+4$$

$$\phi = z + x^2 + y^2 - 4 = 0$$

$$\vec{\nabla}\phi \perp S \quad \Rightarrow \quad \hat{n} = \pm \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$$

$$\vec{\nabla}\phi = 2(x\hat{i}+y\hat{j})+\hat{k} \quad \Rightarrow \text{Choose } +$$

$$|\vec{\nabla}\phi| = \sqrt{4(x^2+y^2)+1} = \sqrt{4r^2+1}$$

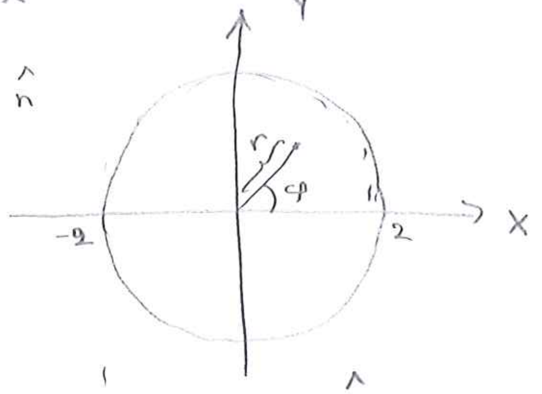


$$x = r\cos\varphi, \quad y = r\sin\varphi$$

$$\vec{\nabla}\times\vec{A}\cdot\hat{n} = (1-x^2)\hat{k}\cdot\left[\frac{2(x\hat{i}+y\hat{j})+\hat{k}}{\sqrt{4r^2+1}}\right]$$

$\hat{n}$

$$= \frac{1-x^2}{\sqrt{4r^2+1}} = \frac{1-r^2\cos^2\varphi}{\sqrt{4r^2+1}}$$

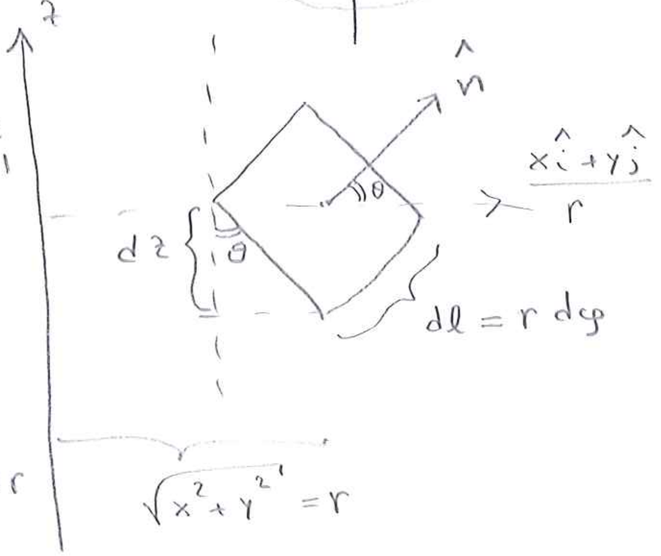


$$d\sigma = \left|\frac{dz}{\cos\theta}\right|\cdot dl$$

$$dz = d(-r^2+4)' = -2r\,dr$$

$$\cos\theta = \hat{n}\cdot\frac{x\hat{i}+y\hat{j}}{r} = \frac{2r^2}{r\sqrt{4r^2+1}} = \frac{2r}{\sqrt{4r^2+1}}$$

$$\Rightarrow d\sigma = \left|\frac{-2r\,dr}{\frac{2r}{\sqrt{4r^2+1}}}\right|\cdot r\,d\varphi = r\sqrt{4r^2+1}\,d\varphi$$



$$\Downarrow$$

$$I = \int_0^2\int_0^{2\pi} \frac{1-r^2\cos^2\varphi}{\sqrt{4r^2+1}}\cdot r\sqrt{4r^2+1}\,d\varphi\,dr$$

$$= \int_0^2\int_0^{2\pi} r(1-r^2\underbrace{\cos^2\varphi}_{\frac{1+\cos 2\varphi}{2}})\,d\varphi\,dr$$

$$= \int_0^2 (2\pi r - r^3\cdot\pi)\,dr = \pi\left(r^2 - \frac{r^4}{4}\right)\Big|_0^2$$

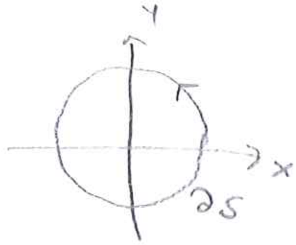
$$= \pi\left(4 - \frac{16}{4} - 0 + 0\right)$$

$$= 0$$

2.c (10 points) Evaluate $I := \iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, d\sigma$ using the Stokes theorem.

$$I = \int_{\partial S} \vec{A} \cdot d\vec{l} \qquad \partial S = \{(x, y, 0) \mid x^2 + y^2 = 4\}$$

$$= \int_{\partial S} (x^2 y\, dx + x\, dy)$$



$x = 2\cos\theta$

$y = 2\sin\theta$

$dx = -2\sin\theta\, d\theta$

$dy = 2\cos\theta\, d\theta$

$$= \int_0^{2\pi} [8\cos^2\theta \sin\theta(-2\sin\theta\, d\theta) + 2\cos\theta(2\cos\theta\, d\theta)]$$

$$= \int_0^{2\pi} [\underbrace{-16\cos^2\theta \sin^2\theta}_{-4\underbrace{\sin^2 2\theta}_{\left(\frac{1-\cos 4\theta}{2}\right)}} + 4\underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}]\, d\theta$$

$$= -2\int_0^{2\pi} (1 - \cos 4\theta)\, d\theta + 2\int_0^{2\pi} (1 + \cos 2\theta)\, d\theta$$

$$= -2(2\pi - 2\pi) = 0.$$

Problem 3 Let $u(x,y) := x^4 + \alpha x^2 y^2 + \beta y^4$ where α and β are real parameters.

3.a (5 points) Find α and β such that u solves Laplace's equation: $\nabla^2 u = 0$.

$u_x = 4x^3 + 2\alpha x y^2$, $u_{xx} = 12x^2 + 2\alpha y^2$

$u_y = 2\alpha x^2 y + 4\beta y^3$, $u_{yy} = 2\alpha x^2 + 12\beta y^2$

$\Rightarrow \nabla^2 u = (12 + 2\alpha) x^2 + (12\beta + 2\alpha) y^2$

$\nabla^2 u = 0 \Rightarrow \boxed{\alpha = -6 \ \& \ \beta = 1}$

3.b (10 points) Find the most general $v(x,y)$ such that $u(x,y)$ and $v(x,y)$ are respectively the real and imaginary parts of $f(x+iy)$ for some entire functions $f : \mathbb{C} \to \mathbb{C}$.

$u = x^4 - 6x^2y^2 + y^4$

$u_x = 4x^3 - 12xy^2$, $u_y = -12x^2y + 4y^3$

$v_y = u_x \Rightarrow v = \int u_x \, dy$

$\Rightarrow v(x,y) = 4x^3 y - 4xy^3 + g(x)$

$\Rightarrow v_x = 12x^2 y - 4y^3 + g'(x)$

$v_x = -u_y \Rightarrow 12x^2y - 4y^3 + g'(x) = -(-12x^2y + 4y^3)$

$\Rightarrow g'(x) = 0 \Rightarrow g(x) = c$ = constant

$\Rightarrow \boxed{v(x,y) = 4x^3y - 4xy^3 + c}$

3.c (10 points) Give an explicit expression for the most general entire functions $f(z)$ such that $u(x, y)$ is the real part of $f(x + iy)$.

$$f(x+iy) = u(x,y) + i v(x,y)$$
$$= x^4 - 6x^2y^2 + y^4 + i(4x^3y - 4xy^3 + c)$$

$\Downarrow$

$$f(x) = x^4 + ic$$

$\Downarrow$

$$\boxed{f(z) = z^4 + ic}$$

we can also check

$$z^4 = (x+iy)^4 = x^4 + 3x^3(iy) + 6x^2(iy)^2 + 3x(iy)^3 + (iy)^4$$
$$= x^4 - 6x^2y^2 + y^4 + i(3x^3y - 3xy^3)$$

$$\Rightarrow \quad f(z) = z^4 + ic \checkmark$$

Problem 4 (15 points) Suppose that $D_r(a)$ is a disc of radius r centered at $a \in \mathbb{C}$, i.e., $D_r(a) := \{z \in \mathbb{C} \mid |z-a| \le r\}$, $f, g : \mathbb{C} \to \mathbb{C}$ are functions such that f is holomorphic in $D_r(a)$ and

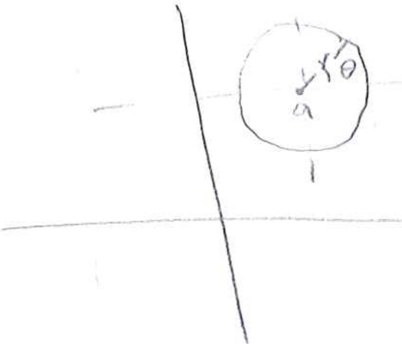
$$g(z) = f(z) + \frac{b}{(z-a)^n},$$

for some nonzero complex number b and some positive integer n. Without using the residue theorem or the formulas for the coefficients of the Laurent's series, show that for any contour C in $D_r(\epsilon)$,

$$\oint_C g(z)dz = \begin{cases} \overset{2\pi i}{b} & \text{if } n = 1, \\ 0 & \text{if } n \ge 2. \end{cases}$$

$\oint_C f(z)\,dz = 0$ because f is holomorphic in $D_r(a)$.

$$\Rightarrow \oint_C g(z)\,dz = \oint_C \frac{b\,dz}{(z-a)^n} = b \oint_C \frac{dz}{(z-a)^n}$$



On C: $z = a + re^{i\theta}$

$dz = ire^{i\theta}d\theta$

$$\Rightarrow \oint_C g(z)\,dz = b\int_0^{2\pi} \frac{ire^{i\theta}\,d\theta}{(re^{i\theta})^n}$$

$$= \frac{ib}{r^{n-1}} \int_0^{2\pi} e^{i(1-n)\theta}\,d\theta$$

$$= \frac{ib}{r^{n-1}} \int_0^{2\pi} \left(\cos[(1-n)\theta] + i\sin[(1-n)\theta]\right)d\theta$$

<u>For $n = 1$</u>: $\oint_C g(z)\,dz = ib\int_0^{2\pi} d\theta = 2\pi i b$

<u>For $n \geq 2$</u>: $\oint_C g(z)\,dz = \left(\frac{\sin(1-n)\theta}{1-n} - \frac{i\cos(1-n)\theta}{1-n}\right)\Big|_0^{2\pi}$

$= 0$

Problem 5 Let $f(z) := \dfrac{\sqrt{e} - e^{-z^2}}{z(2z^2+1)^2}$ and

$$S := \left\{ z \in \mathbb{C} \mid \mathrm{Re}(z)^4 + \mathrm{Im}(z)^4 \leq 1 \text{ and } \mathrm{Im}(z) \leq \frac{1}{2} \right\}.$$

5.a (10 points) Find the singularities of $f(z)$ in S and determine if they are essential singularities or poles. If any one of them is a pole, determine its order.

$f(z)$ is the ratio of two entire functions so its singularities are poles the coincide with zeros of $z(2z^2+1)^2$.

$2z^2+1=0 \Rightarrow \boxed{z = \pm \frac{i}{\sqrt{2}}}$ $\qquad f(z) = \dfrac{\sqrt{e} - e^{-z^2}}{4z(z+\frac{i}{\sqrt{2}})^2(z-\frac{i}{\sqrt{2}})^2}$

$\boxed{z=0}$

$\mathrm{Im}\left(\frac{i}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} > \frac{1}{2} \Rightarrow z = \frac{i}{\sqrt{2}} \notin S$

So singularities belonging to S are $z_0 = 0$, $z_0 = \frac{-i}{\sqrt{2}}$

$I := \lim_{z \to 0} z f(z) = \lim_{z\to 0} \dfrac{\sqrt{e} - e^{-z^2}}{(2z^2+1)^2} = \dfrac{\sqrt{e}-1}{1} \Rightarrow$ $\boxed{z_0 = 0 \text{ is a simple pole (order} = 1).}$

& $\boxed{R(0) = \sqrt{e} - 1}$

$I := \lim_{z \to -\frac{i}{\sqrt{2}}} (z + \frac{i}{\sqrt{2}}) f(z) = \lim_{z \to -\frac{i}{\sqrt{2}}} \dfrac{\sqrt{e} - e^{-z^2}}{4z(z+\frac{i}{\sqrt{2}})(z-\frac{i}{\sqrt{2}})^2}$

$\left(e^{-(-\frac{i}{\sqrt{2}})^2} = e^{-(-\frac{1}{2})} = \sqrt{e} \quad \text{so} \quad \sqrt{e} - e^{-(-\frac{i}{\sqrt{2}})^2} = 0 \right)$

$\Rightarrow I = \lim_{z \to -\frac{i}{\sqrt{2}}} \dfrac{2ze^{-z^2}}{4\left[\underbrace{(z+\frac{i}{\sqrt{2}})(z-\frac{i}{\sqrt{2}})^2}_{0} + z(z-\frac{i}{\sqrt{2}})^2 + \underbrace{2z(z+\frac{i}{\sqrt{2}})(z-\frac{i}{\sqrt{2}})}_{0}\right]}$

$= \dfrac{2(-\frac{i}{\sqrt{2}})\sqrt{e}}{4(-\frac{i}{\sqrt{2}})(-\frac{2i}{\sqrt{2}})^2} = \dfrac{-\sqrt{e}}{4} \Rightarrow$ $\boxed{z_0 = \frac{-i}{\sqrt{2}} \text{ is also a simple pole \& } R(-\frac{i}{\sqrt{2}}) = \frac{-\sqrt{e}}{4}.}$

5.b (15 points) Evaluate $I := \oint_C f(z)dz$ where C be the counterclockwise oriented boundary of S.

We showed that S includes a pair of singularities namely $z_0 = 0$ & $z_0 = -\frac{i}{\sqrt{2}}$ which are simple poles with residue $R(0) = \sqrt{e} - 1$ and $R(-\frac{i}{\sqrt{2}}) = \frac{\sqrt{e}}{4}$.

Therefore $\boxed{I = 2\pi i \left[R(0) + R(-\frac{i}{\sqrt{2}}) \right]}$

$$= 2\pi i \left(\sqrt{e} - 1 - \frac{\sqrt{e}}{4} \right)$$

$$= \boxed{2\pi i \left(\frac{3}{4}\sqrt{e} - 1 \right)}$$