Math 303: Quiz # 3

Fall 2013, 50 minutes

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name	Student ID Number	Signature

---

1 (7 points) Let $f : \mathbb{C} \to \mathbb{C}$, $u : \mathbb{R}^2 \to \mathbb{R}$, and $v : \mathbb{R}^2 \to \mathbb{R}$ be functions such that for all $x, y \in \mathbb{R}$, $f(x+iy) = u(x,y) + iv(x,y)$. Given that $u(x,y) = xy$, find the general form of $v(x,y)$ such that f is an entire function, i.e., it is differentiable at every $z \in \mathbb{C}$.

$$u_x = v_y \Rightarrow y = v_y \Rightarrow v = \frac{y^2}{2} + f(x)$$

$$u_y = -v_x \Rightarrow x = -f'(x) \Rightarrow f(x) = -\frac{x^2}{2} + c$$

const.

$$\boxed{v(x,y) = \frac{y^2 - x^2}{2} + c}$$

Bonus (5 pb):

$$f(x+iy) = xy + i\left[\frac{y^2 - x^2}{2} + c\right]$$

$$= \frac{i}{2}\left[y^2 - x^2 - 2ixy\right] + ic$$

$$= -\frac{i}{2}\underbrace{(x^2 - y^2 + 2ixy)}_{(x+iy)^2} + ic \Rightarrow$$

$$\Rightarrow \boxed{f(z) = -\frac{i}{2}z^2 + ic}$$

when c is som complex number

2 (10 points) Find the singularities of the function $f(z) := \dfrac{\cos z}{(z-\pi/2)^4}$, determine if they are essential singularities or poles, and for each pole find both its order and the residue of the function.

There is a single singularity, i.e., $z_0 = \frac{\pi}{2}$, because.

$$\lim_{z \to \frac{\pi}{2}} f(z) = \lim_{z \to \frac{\pi}{2}} \frac{-\sin z}{4(z-\frac{\pi}{2})^3} = \infty$$

For $z \neq z_0$ both $\cos z$ & $\dfrac{1}{(z-\frac{\pi}{2})^4}$ are differentiable.

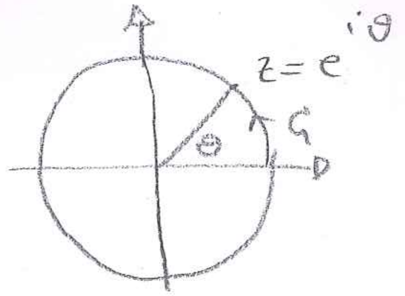
$$f(z) = \frac{\sin(z-\frac{\pi}{2})}{(z-\frac{\pi}{2})^4} = \frac{1}{(z-\frac{\pi}{2})^4}\Big[(z-\tfrac{\pi}{2}) - \frac{1}{3!}(z-\tfrac{\pi}{2})^3 + \frac{1}{5!}(z-\tfrac{\pi}{2})^5 - \frac{1}{7!}(z-\tfrac{\pi}{2})^7 \pm \cdots\Big]$$

$$= \underbrace{\frac{1}{(z-\frac{\pi}{2})^3} - \frac{1}{3!\,(z-\frac{\pi}{2})}}_{\text{principle part}} + \underbrace{\frac{(z-\frac{\pi}{2})}{5!} - \frac{1}{7!}(z-\tfrac{\pi}{2})^3 \pm \cdots}_{\text{analytic part}}$$

So $z_0 = \frac{\pi}{2}$ is a pole of order 3.

and $\mathrm{Res}(\frac{\pi}{2}) = -\dfrac{1}{3!} = -\dfrac{1}{6}$.

3 (13 points) Evaluate $\displaystyle\int_0^{2\pi} \frac{\cos\theta \, d\theta}{5 - 4\cos\theta}$.



$$z = e^{i\theta} \Rightarrow dz = iz\, d\theta \Rightarrow d\theta = -\frac{i\,dz}{z}$$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \quad \Rightarrow$$

$$I := \int_0^{2\pi} \frac{\cos\theta\, d\theta}{5 - 4\cos\theta} = \oint_C \left(\frac{\frac{1}{2}\left(z + \frac{1}{z}\right)}{5 - 2\left(z + \frac{1}{z}\right)} \right)\left(\frac{-i\,dz}{z}\right)$$

$$= \frac{-i}{2} \oint_C \frac{(z^2+1)\,dz}{z(5z - 2z^2 - 2)} = \frac{i}{2} \oint_C \frac{(z^2+1)\,dz}{z(2z^2 - 5z + 2)}$$

$$f(z) = \frac{i}{2}\left[\frac{z^2+1}{z(2z^2 - 5z + 2)}\right] \quad \text{has 3 simple poles}$$

$$z_0 = 0, \quad z_\pm = \frac{5 \pm \sqrt{25 - 16}}{4} = \begin{cases} 2 =: z_+ \\ \frac{1}{2} =: z_- \end{cases}$$

Only z_0 & z_- are enclosed by C

$$f(z) = \frac{i}{2}\left[\frac{z^2+1}{2z(z - z_-)(z - z_+)}\right]$$

$$Re(0) = \lim_{z \to 0} z f(z) = \frac{i}{4(-z_-)(-z_+)} = \frac{i}{4 z_- z_+} = \frac{i}{4}$$

$$Re(z_-) = \lim_{z \to z_-} (z - z_-) f(z) = \frac{i}{2}\left[\frac{z_-^2 + 1}{2z_-(z_- - z_+)}\right]$$

$$= \frac{i}{2}\left[\frac{\frac{1}{4} + 1}{1\underbrace{\left(\frac{1}{2} - 2\right)}_{-\frac{3}{2}}}\right] = -\frac{5i}{12}$$

$$\Rightarrow \boxed{I = 2\pi i\left[Re(0) + Re(z_-)\right]} = 2\pi i\left[\frac{i}{4} - \frac{5i}{12}\right]$$

$$= (2\pi i)\left(\frac{i}{4}\right)\underbrace{\left(1 - \frac{5}{3}\right)}_{-\frac{2}{3}} = \boxed{\frac{\pi}{3}}$$