Math 303: Quiz # 2

Fall 2013

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 45 minutes.
- **Give details of your response to each problem.**

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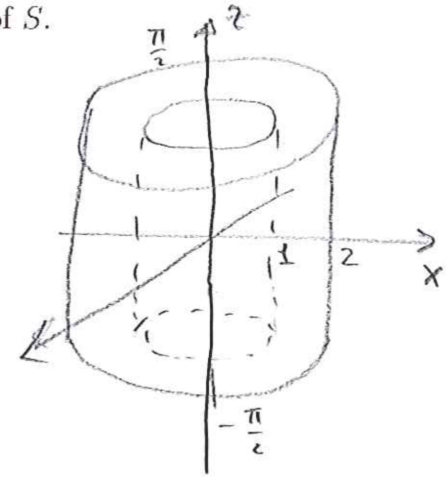
1 (8 points) Let D be the cylindrical shell defined in cylindrical coordinates (ρ, θ, z) by $1 \le \rho \le 2$ and $-\frac{\pi}{2} \le z \le \frac{\pi}{2}$, and S be the boundary of D. Use Divergence theorem to compute the flux Φ of the vector field $\mathbf{F} = \cos z\, \mathbf{e}_\rho + \sin z\, \mathbf{k}$ through S, where $\mathbf{e}_\rho$ is the unit vector along the radial coordinate ρ and $\mathbf{k}$ is the unit vector along the positive z-axis. Recall that $\Phi := \int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{n}$ is the unit normal outward vector to S and $d\sigma$ is the surface element of S.

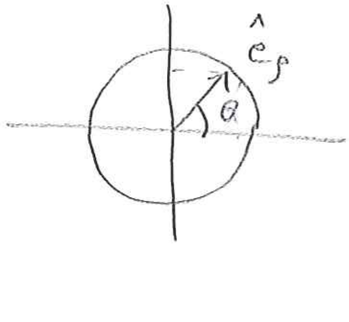


According to the Divergence thm:

$$\Phi = \iiint_V \vec{\nabla} \cdot \vec{F}\, dV$$

$$e_\rho = \cos\theta\, \hat{i} + \sin\theta\, \hat{j} = \frac{x}{\sqrt{x^2+y^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2}}\hat{j}$$

$$\Rightarrow \vec{F} = \frac{\cos(z)x}{\sqrt{x^2+y^2}}\hat{i} + \frac{\cos(z)y}{\sqrt{x^2+y^2}}\hat{j} + \sin z\, \hat{k}$$



$$\vec{\nabla} \cdot \vec{F} = \cos z\left[\frac{1}{\sqrt{x^2+y^2}} - \frac{1}{2}(x^2+y^2)^{-3/2} x(2x)\right] + \cos z\left[\frac{1}{\sqrt{x^2+y^2}} - \frac{1}{2}(x^2+y^2)^{-3/2} y(2y)\right] + \cos z$$

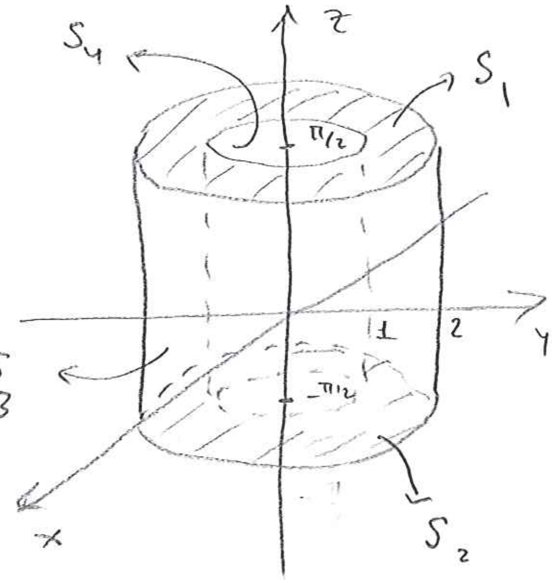
$$= \cos z\left[\frac{1}{\rho} - \frac{x^2}{\rho^3} + \frac{1}{\rho} - \frac{y^2}{\rho^3} + 1\right]$$

$$= \cos z\left[\frac{2}{\rho} - \frac{\rho^2}{\rho^3} + 1\right] = \cos z\left(1 + \frac{1}{\rho}\right)$$

$$\Phi = \int_{-\frac{\pi}{2}}^{\pi/2}\int_0^{2\pi}\int_1^2 \cos(z)\left(1+\frac{1}{\rho}\right)\rho\, d\rho\, d\theta\, dz = \left(\sin(z)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right)(2\pi)\left(\frac{\rho^2}{2} + \rho\right)\Big|_1^2$$

$$= 4\pi\left(4 - \frac{3}{2}\right) = 10\pi$$

2 (10 points) Compute the flux Φ of Problem 1 without using the Divergence theorem, i.e., evaluate it by performing the surface integral $\int_S \mathbf{F} \cdot d\mathbf{S}$ directly.



Top face (S_1): $z = \frac{\pi}{2} \Rightarrow \vec{F} = \hat{k} = \hat{n}$

$\Rightarrow \vec{F} \cdot \hat{n} = 1 \Rightarrow$

$$\Phi_1 := \int_{S_1} \vec{F} \cdot \hat{n}\, d\sigma = 1 \int_{S_1} d\sigma = \text{Area of } S_1$$

$$= \pi(2^2 - 1^2) = 3\pi$$

Bottom face (S_2): $z = -\frac{\pi}{2} \Rightarrow \vec{F} = -\hat{k} = \hat{n}$

$$\Rightarrow \vec{F} \cdot \hat{n} = 1 \Rightarrow \Phi_2 := \int_{S_2} \vec{F} \cdot \hat{n}\, d\sigma = 3\pi$$

Outer cylindrical boundary (S_3): $\rho = 2$, $\hat{n} = \hat{e}_\rho \Rightarrow$

$$\vec{F} \cdot \hat{n} = \cos z \Rightarrow \Phi_3 := \int_{S_3} \vec{F} \cdot \hat{n}\, d\sigma = \int_{-\frac{\pi}{2}}^{\pi/2} \int_0^{2\pi} \cos z \; 2\, d\theta\, dz$$

$$= 2(2\pi)\underbrace{\left(\sin z \Big|_{-\pi/2}^{\pi/2}\right)}_{2} = 8\pi$$

Inner cylindrical bounds (S_4): $\rho = 1$, $\hat{n} = -\hat{e}_\rho \Rightarrow$

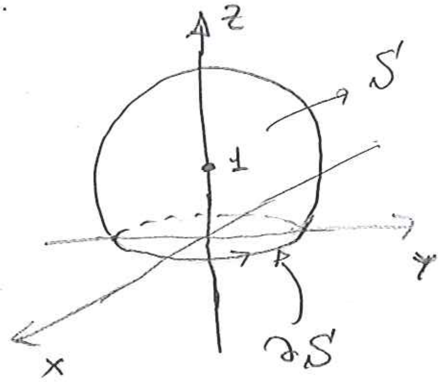
$$\vec{F} \cdot \hat{n} = -\cos z \Rightarrow \Phi_4 := \int_{S_4} \vec{F} \cdot \hat{n}\, d\sigma = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} (-\cos z)\, 1\, d\theta\, dz$$

$$= -1(2\pi)\underbrace{\left(\sin z \Big|_{-\pi/2}^{\pi/2}\right)}_{2} = -4\pi$$

So $\Phi = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 = 3\pi + 3\pi + 8\pi - 4\pi$

$= 10\pi$ ✓

3 (12 points) Let S be the part of the sphere defined by $(z-1)^2+x^2+y^2=4$ that lies lines above the x-y plane, i.e., $S := \left\{(x,y,z)\in\mathbb{R}^3 \mid z = 1+\sqrt{4-x^2-y^2}\right\}$, and $\mathbf{F}:\mathbb{R}^3\to\mathbb{R}^3$ be the vector field defined by $\mathbf{F}(x,y,z)=\sin(xz)\,\mathbf{i}+xy\,\mathbf{j}+xe^z\,\mathbf{k}$, where $\mathbf{i},\mathbf{j}$, and $\mathbf{k}$ are unit vectors along the positive x-, y-, and z-axes respectively. Calculate the Flux of $\nabla\times\mathbf{F}$ through S.



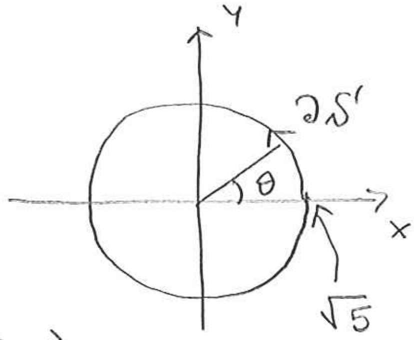
By Stokes' thm

$$\Phi := \int_S \vec\nabla\times\vec F\cdot\hat n\, d\sigma = \int_{\partial S'} \vec F\cdot d\vec x$$

∂S := boundary of S is given by $1+\sqrt{4-(x^2+y^2)} = z = 0$

$\Rightarrow \quad 4-(x^2+y^2) = -1$

$\Rightarrow \quad \boxed{x^2+y^2 = 5}$



In polar coordinates $\rho=\sqrt5$, $\theta\in[0,2\pi)$

On ∂S; $z=0 \Rightarrow F = xy\vec j + x\vec k$

$d\vec x\cdot\vec k = 0 \Rightarrow \vec F\cdot d\vec x = xy\,dy$

$x = \sqrt5\cos\theta$

$y = \sqrt5\sin\theta \Rightarrow dy = \sqrt5\cos\theta\, d\theta \Rightarrow$

$$\boxed{\Phi = \int_0^{2\pi} (\sqrt5)^3 \sin\theta\cos^2\theta\, d\theta = -25\sqrt5\int_1^1 u^2\,du = 0}$$

let $u := \cos\theta \Rightarrow du = -\sin\theta\, d\theta$