Math 303, Fall 2013
Midterm Exam 2

Problem 1.

1.a (5 points) Give the definition and an example of an entire function $f : \mathbb{C} \to \mathbb{C}$.

f is entire if it is analytic at every $z \in \mathbb{C}$.

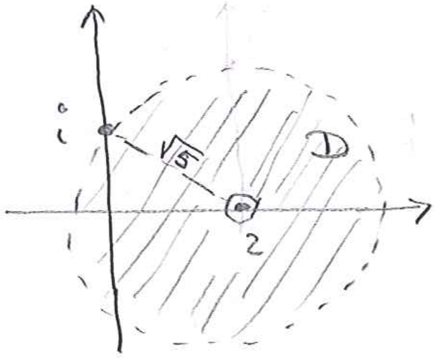
1.b (5 points) Give the statement of Cauchy's theorem on analytic functions.

If $f : \mathbb{C} \to \mathbb{C}$ is analytic on and inside a contour C,
then $\oint_C f(z)\,dz = 0$.

1.c (5 points) Give the definition of a pole of a complex-valued function f. A pole z_0 is a singularity of f when the principal part of the Laurent series exapansion of f about z_0 has finitely many nonzero terms.

1.d (5 points) Give the definition of residue of a complex-valued function f at a point $z_0 \in \mathbb{C}$ and state the Residue Theorem. The residue at z_0 is the coefficient of terms $\frac{1}{z - z_0}$ in the Laurent Series expansion of f about z_0.

Problem 2 (20 points) Find the Laurent series expansion for $f(z) = \dfrac{1}{(z-i)(z-2)}$ about $z = 2$. Determine the largest region within which this series converges to $f(z)$.



$$\frac{1}{z-i} = \frac{1}{z-2+2-i} = \frac{1}{(2-i)\left(1-\frac{2-z}{2-i}\right)}$$

$$w := \frac{2-z}{2-i} \Rightarrow |w| = \frac{|2-z|}{\sqrt{5}}$$

in region D $\quad |w| < 1 \Rightarrow$

$$\frac{1}{1-w} = \sum_{k=0}^{\infty} w^k$$

$$\Rightarrow \frac{1}{z-i} = \frac{1}{(2-i)}\frac{1}{1-w} = \frac{1}{2-i}\sum_{k=0}^{\infty}\left(\frac{2-z}{2-i}\right)^k$$

$$= \sum_{k=0}^{\infty} \frac{(z-2)^k(-1)^k}{(2-i)^{k+1}}$$

$$\Rightarrow f(z) = \frac{1}{z-2}\sum_{k=0}^{\infty}\frac{(z-2)^k(-1)^k}{(2-i)^{k+1}}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k(z-2)^{k-1}}{(2-i)^{k+1}}$$

$$= \frac{1}{(2-i)(z-2)} + \sum_{k=1}^{\infty}\frac{(-1)^k(z-2)^{k-1}}{(2-i)^{k+1}} \qquad n = k-1$$

$$= \frac{2+i}{5(z-2)} + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(z-2)^n}{(2-i)^{n+2}} \qquad \forall z \in D$$

$$= \frac{b_1}{z-2} + \sum_{n=0}^{\infty} a_n (z-2)^n$$

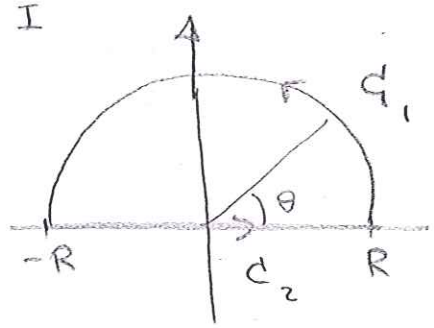
when $b_1 := \dfrac{2+i}{5}$ and $\forall n \geq 0,\ a_n := \dfrac{(-1)^{n+1}}{(2-i)^{n+2}}$

$$D = \{ z \in \mathbb{C} \mid 0 < |z-2| < \underbrace{\sqrt{5}}_{|2-i|} \}$$

Problem 3 (25 points) Use Residue theorem to evaluate $\underbrace{\int_0^\infty \frac{dx}{1+x^4}}_{I}$.

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$$

$$= \frac{1}{2}\left[\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{1+x^4}\right]$$



$$= \frac{1}{2}\left[\lim_{R\to\infty}\left(\oint_{C} \frac{dz}{1+z^4} - \int_{C_1} \frac{dz}{1+z^4}\right)\right]$$

$C = C_1 \cup C_2$

on C_1: $z = Re^{i\theta} \Rightarrow dz = iRe^{i\theta}d\theta \Rightarrow$

$$\lim_{R\to\infty} \int_{C_1} \frac{dz}{1+z^4} = \lim_{R\to\infty} \int_0^{\pi} \frac{iRe^{i\theta}d\theta}{1+R^4e^{4i\theta}}$$

$$= \lim_{R\to\infty} \frac{i}{R^3}\int_0^{\pi} \frac{e^{i\theta}d\theta}{e^{4i\theta}} = 0$$

$$\Rightarrow I = \frac{1}{2}\lim_{R\to\infty}\oint_{C} \frac{dz}{1+z^4} = \frac{1}{2}\lim_{R\to\infty}\left[2\pi i \sum_{i=1}^{N} \operatorname{Res}(z_i)\right]$$

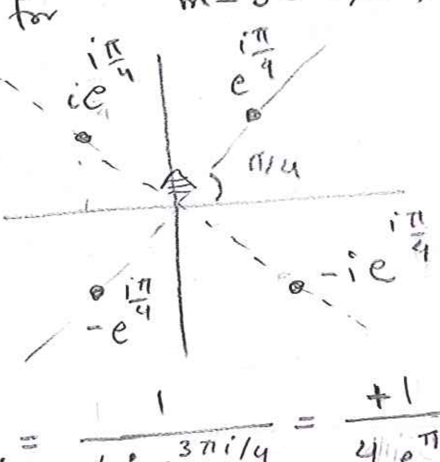
z_i are singularities of $\frac{1}{1+z^4}$ enclosed by C and N is their number.

$1+z^4 = 0 \Rightarrow z^4 = -1 = e^{i\pi + (2\pi i m)}$ for $m \in \mathbb{Z}$

$$\Rightarrow z = e^{\frac{i\pi}{4}} e^{\frac{\pi i m}{2}} = \begin{cases} e^{i\pi/4} & \text{for } m = 0, \pm 4, \pm 8, \dots \\ ie^{\frac{i\pi}{4}} & \text{for } m-1 = 0, \pm 4, \pm 8, \dots \\ -e^{i\pi/4} & \text{for } m-2 = 0, \pm 4, \pm 8, \dots \\ -ie^{\frac{i\pi}{4}} & \text{for } m-3 = 0, \pm 4, \pm 8, \dots \end{cases}$$

So

$$I = \frac{1}{2}(2\pi i)\left[\operatorname{Res}(e^{\frac{i\pi}{4}}) + \operatorname{Res}(ie^{\frac{i\pi}{4}})\right]$$



$$f(z) = \frac{1}{1+z^4} = \frac{1}{(z-e^{i\pi/4})(z-ie^{i\pi/4})(z+e^{\frac{i\pi}{4}})(z+ie^{\frac{i\pi}{4}})}$$

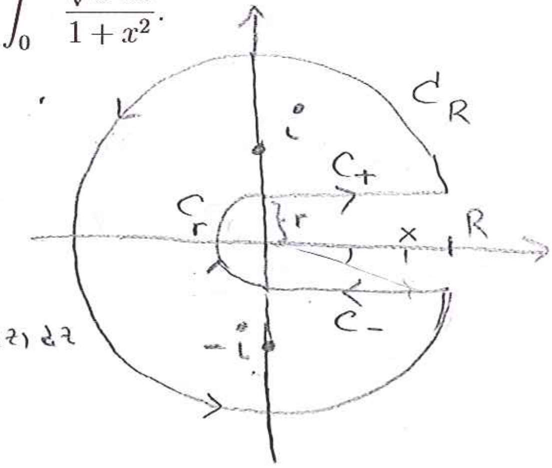
$$\operatorname{Res}(e^{\frac{i\pi}{4}}) = \frac{1}{(e^{\frac{i\pi}{4}}-ie^{\frac{i\pi}{4}})(e^{i\pi/4}+e^{i\pi/4})(e^{\frac{i\pi}{4}}+ie^{\frac{i\pi}{4}})}$$

$$= \frac{1}{(1-i)\,2(1+i)\,e^{3i\pi/4}} = \frac{1}{4ie^{\frac{\pi i}{4}}}$$

$$\operatorname{Res}(ie^{\frac{i\pi}{4}}) = \frac{1}{(ie^{i\pi/4}-e^{i\pi/4})(ie^{i\pi/4}+e^{i\pi/4})(ie^{i\pi/4}+ie^{i\pi/4})} = \frac{1}{-4i\,e^{3\pi i/4}} = \frac{+1}{4e^{\pi i/4}}$$

$\Downarrow$

Problem 4. (25 points) Use Residue theorem to evaluate $\overbrace{\int_0^\infty \frac{\sqrt{x}\,dx}{1+x^2}}^{I}$.



$C = C_+ \cup C_R \cup C_- \cup C_r$

$I = \lim_{\substack{R\to\infty \\ r\to 0}} \int_{C_+} \frac{\sqrt{z}\,dz}{1+z^2}$ $\qquad f(z) := \frac{\sqrt{z}}{1+z^2}$

$\oint_C f(z)\,dz = \int_{C_+} f(z)\,dz + \int_{C_R} f(z)\,dz + \int_{C_-} f(z)\,dz + \int_{C_r} f(z)\,dz$

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on C_R in the limit $R\to+\infty$:

$z = Re^{i\theta} \qquad dz = iRe^{i\theta}d\theta$

$I_R := \int_{C_R} f(z)\,dz = \int_{\epsilon}^{2\pi-\epsilon} \frac{\sqrt{R}\,e^{i\theta/2}\, iRe^{i\theta}\,d\theta}{1+R^2e^{2i\theta}} \qquad \epsilon := \tan^{-1}\left(\frac{r}{R}\right)$

in the limit $\begin{matrix} r\to 0 \\ R\to\infty \end{matrix}$, $I_R \to 0$.

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on C_- : $z = x - ir = \sqrt{x^2+r^2}\, e^{i[2\pi - \tan^{-1}(\frac{r}{x})]} \qquad dz = dx$

$I_- := \int_{C_-} f(z)\,dz = \int_R^0 \frac{(x^2+r^2)^{\frac{1}{4}}\, e^{i(\pi - \frac{1}{2}\tan^{-1}(\frac{r}{x}))}}{1+(x-ir)^2}\,dx$

in the limit $\left.\begin{matrix} r\to 0 \\ R\to\infty \end{matrix}\right\}$, $I_- \to -\int_\infty^0 \frac{\sqrt{x}\,dx}{1+x^2} = \int_0^\infty \frac{\sqrt{x}\,dx}{1+x^2} = I$

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on C_r' : $z = re^{i\theta} \qquad dz = ire^{i\theta}d\theta$

$I_r := \int_{C_r} f(z)\,dz = \int_{\frac{3\pi}{2}}^{\pi/2} \left(\frac{\sqrt{r}\,e^{i\theta/2}}{1+r^2e^{2i\theta}}\right) ire^{i\theta}\,d\theta \xrightarrow{\text{as } r\to 0} 0$

$\lim_{R\to\infty} \oint_C f(z)\,dz = 2\pi i\,(\mathrm{Res}(i) + \mathrm{Res}(-i)) \qquad f(z) = \frac{\sqrt{z}}{(z+i)(z-i)}$

$\mathrm{Res}(i) = \lim_{z\to i} \frac{\sqrt{z}}{z+i} = \frac{\sqrt{i}}{2i} = \frac{1}{2\sqrt{i}}$

$\mathrm{Res}(-i) = \lim_{z\to -i} \frac{\sqrt{z}}{z-i} = \frac{\sqrt{-i}}{-2i} = \frac{1}{2\sqrt{-i}} = \frac{1}{2i\sqrt{i}} = -\frac{i}{2\sqrt{i}}$

$\Rightarrow \lim_{R\to\infty} \oint_C f(z)\,dz = 2\pi i\left(\frac{1}{2\sqrt{i}} - \frac{i}{2\sqrt{i}}\right) = \frac{\pi}{\sqrt{i}}(1+i) = \pi e^{-\frac{\pi i}{4}}(1+i)$

$= \pi\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)(1+i) = \frac{\pi\sqrt{2}}{2}(1-i)(1+i) = \sqrt{2}\,\pi$

$\Rightarrow \sqrt{2}\,\pi = 2I \quad \Rightarrow \quad \boxed{I = \frac{\sqrt{2}}{2}\pi}$

Soln of Problem 3 continues:

$$I = \pi i \left[\frac{-i}{4e^{i\pi/4}} + \frac{1}{4e^{i\pi/4}} \right]$$

$$= \frac{\pi}{4} e^{-i\pi/4} (1+i)$$

$$e^{-i\frac{\pi}{4}} = \cos(\tfrac{\pi}{4}) - i\sin(\tfrac{\pi}{4})$$
$$= \frac{\sqrt{2}}{2}(1-i)$$

$$= \left(\frac{\pi}{4}\right)\left(\frac{\sqrt{2}}{2}\right)\underbrace{(1-i)(1+i)}_{2}$$

$$= \frac{\pi\sqrt{2}}{4}$$

Problem 5. (10 points) Evaluate $\int_0^{\pi} \delta(4x^2-1)\sin(\pi x)dx$, where δ denotes Dirac's delta function.

$y := 4x^2 - 1$ $\qquad x = \frac{1}{2}\sqrt{y+1}$ for $x \in [0, \pi]$

$dy = 8x\,dx \Rightarrow \qquad dx = \frac{dy}{8\left(\frac{1}{2}\sqrt{y+1}\right)} = \frac{dy}{4\sqrt{y+1}}$

$$\Rightarrow \int_0^{\pi} \delta(4x^2-1)\sin(\pi x)\,dx = \int_{-1}^{4\pi^2-1} \delta(y)\sin\left[\frac{\pi}{2}\sqrt{y+1}\right]\frac{dy}{4\sqrt{y+1}}$$

$$= \frac{\sin\left(\frac{\pi}{2}\right)}{4} = \frac{1}{4}$$