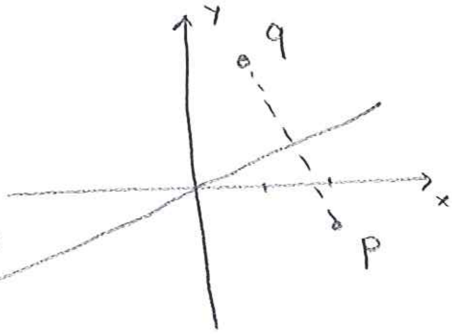
Math 303 - Fall 2013
Midterm Exam 1

Problem 1. Let L be the line in the plane $\mathbb{R}^2$ that passes through the origin and makes an angle $\frac{\pi}{8}$ with the positive x-axis, p be the point in $\mathbb{R}^2$ with Cartesian coordinates $(2,-1)$, and q be the mirror image of p with respect to L.

1.a (10 points) Express p as a complex number and use the algebraic operations corresponding to reflection with respect to the x-axis and rotation in the complex plane to find the coordinates of q.



$p = 2 - i$

$$q = e^{\frac{i\pi}{8}}\left(e^{-\frac{i\pi}{8}}\,p\right)^*$$

$$= e^{\frac{i\pi}{4}}\,p^* = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)(2+i)$$

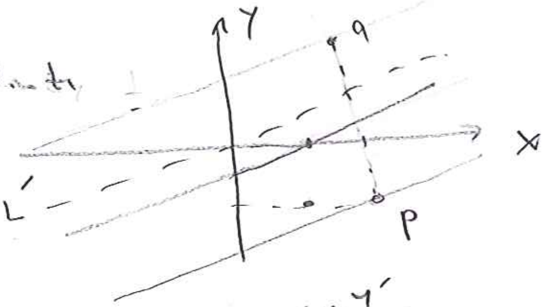
$$= \frac{\sqrt{2}}{2}(1+i)(2+i)$$

$$= \frac{\sqrt{2}}{2}\left[(2-1)+i(2+1)\right]$$

$$= \frac{\sqrt{2}}{2}(1+3i)$$

So $q = \left(\frac{\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$

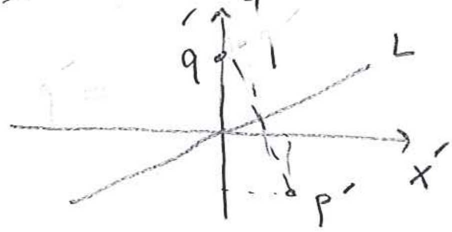
1.b (10 points) Repeat part **1.a** for the case that L passes through the point $(1,0)$ rather than $(0,0)$.



We just need to choose a coordinate system that is centered at $(1,0)$

So we translate

$$\begin{cases} x \to x' = x-1 = 1 \\ y \to y' = y \end{cases} \qquad p \to p' = (1,-1) = 1-i$$



Now find mirror image of p'

Say $q' = e^{i\pi/8}\left(e^{-i\pi/8}\,p'\right)^* = e^{\frac{i\pi}{4}}(1+i)$

$$= \frac{\sqrt{2}}{2}(1+i)(1+i) = \frac{\sqrt{2}}{2}\cdot 2i = \sqrt{2}\,i = (0,\sqrt{2})$$

Now change coordinate back (translate back) to find $q = (1, \sqrt{2})$

Problem 2 (25 points) Let P and S be respectively the plane and sphere that are defined by $x - y + z = 2$ and $x^2 + y^2 + z^2 = 4$ in a Cartesian coordinate system in $\mathbb{R}^3$. Use the method of Lagrange multipliers to find the minimum and maximum value of $x + y - z$ on the intersection of P and S.

① $\phi_1 := x - y + z - 2 = 0$ $\qquad f(x,y,z) = x + y - z$

② $\phi_2 := x^2 + y^2 + z^2 - 4 = 0$

$F = f + \lambda_1 \phi_1 + \lambda_2 \phi_2$

$= x + y - z + \lambda_1 (x - y + z - 2) + \lambda_2 (x^2 + y^2 + z^2 - 4)$

$\frac{\partial F}{\partial x} = 1 + \lambda_1 + 2x\lambda_2 \overset{③}{=} 0$

$\frac{\partial F}{\partial y} = 1 - \lambda_1 + 2y\lambda_2 \overset{④}{=} 0$ $\quad \hookrightarrow 2\lambda_2 (y + z) = 0$

$\frac{\partial F}{\partial z} = -1 + \lambda_1 + 2z\lambda_2 \overset{⑤}{=} 0$

$\lambda_2 \neq 0$ because if $\lambda_2 = 0$ ③ & ④ $\Rightarrow 2 = 0$ ※

$\Downarrow$

⑥ $\boxed{z = -y}$ insert this in ① & ② $\Rightarrow$

$\begin{cases} x - 2y = 2 \\ x^2 + 2y^2 = 4 \end{cases} \Rightarrow \boxed{x = 2(y+1)}$ ⑦

$\hookrightarrow 4(y+1)^2 + 2y^2 = 4$

$2(y+1)^2 + y^2 = 2$

$\Rightarrow 3y^2 + 4y + 2 = 2 \Rightarrow y(3y+4) = 0 \Rightarrow \begin{cases} y = 0 \\ y = -\frac{4}{3} \end{cases}$

$y = 0 \xrightarrow{⑥ \& ⑦} \begin{cases} x = 2 \\ z = 0 \end{cases} \hookrightarrow P_1 = (2, 0, 0) \Rightarrow f(P_1) = 2$

$y = -\frac{4}{3} \xrightarrow{⑥ \& ⑦} \left\{ \begin{array}{l} x = 2(-\frac{1}{3}) = -\frac{2}{3} \\ z = \frac{4}{3} \end{array} \right\} \Rightarrow P_2 = (-\frac{2}{3}, -\frac{4}{3}, \frac{4}{3})$

$f(P_2) = -\frac{2}{3} - \frac{4}{3} - \frac{4}{3} = -\frac{10}{3}$

So minimum value of f is $-\frac{10}{3}$ and its maximum value is 2.

Problem 3 Let $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ be differentiable vector field and

$$\mathbf{C} := \mathbf{A} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{A}, \qquad (1)$$

where $(\mathbf{A} \cdot \nabla)\mathbf{A} := \left[A_1 \frac{\partial}{\partial x} + A_2 \frac{\partial}{\partial y} + A_3 \frac{\partial}{\partial z}\right] \mathbf{A}$, A_1, A_2, A_3 are components of $\mathbf{A}$, i.e., $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, and $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors along the x-, y-, and z-axes.

3.a (10 points) Use the properties of the Kronecker and Levi Civita symbols to express $\mathbf{C}$ as the gradient of a scalar function $f : \mathbb{R}^3 \to \mathbb{R}$, i.e., show that $\mathbf{C} = \nabla f$ for some f and give an explicit formula for f in terms of $\mathbf{A}$.

$$(\vec{A} \times (\vec{\nabla} \times \vec{A}))_i = \sum_{j,k=1}^{3} \epsilon_{ijk} A_j (\vec{\nabla} \times \vec{A})_k$$

$$= \sum_{j,k=1}^{3} \epsilon_{ijk} A_j \sum_{l,m=1}^{3} \epsilon_{klm} \partial_l A_m$$

$$= \sum_{j,k,l,m=1}^{3} \underset{\underset{\epsilon_{kij}}{\parallel}}{\epsilon_{ijk}} \epsilon_{klm} A_j \partial_l A_m$$

$$= \sum_{j,l,m=1}^{3} (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) A_j \partial_l A_m$$

$$= \sum_{j=1}^{3} (A_j \partial_i A_j - A_j \partial_j A_i)$$

$$= \sum_{j=1}^{3} \frac{1}{2} \partial_i A_j^2 - \sum_{j=1}^{3} A_j \partial_j A_i$$

$$= \frac{1}{2} \partial_i \underbrace{\sum_{j=1}^{3} A_j^2}_{|\vec{A}|^2} - \underbrace{\sum_{j=1}^{3} A_j \partial_j A_i}_{(\vec{A} \cdot \vec{\nabla}) A_i}$$

$$\Rightarrow \quad \vec{A} \times (\vec{\nabla} \times \vec{A}) = \frac{1}{2} \vec{\nabla}(|\vec{A}|^2) - (\vec{A} \cdot \vec{\nabla}) \vec{A}$$

$$\overset{\parallel}{\nabla}$$

$$\vec{C} = \frac{1}{2} \vec{\nabla}(|\vec{A}|^2)$$

So $f := \frac{1}{2} |\vec{A}|^2$.

3.b (10 points) Evaluate the right-hand side of (1) and compute $\mathbf{C}(1,2,3)$ for the following choice of $\mathbf{A}$ (Do not use your response to part **3.a.**)

$$\mathbf{A}(x,y,z) := (x-z\,,\,x+y\,,\,x+z). \tag{2}$$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ x-z & x+y & x+z \end{vmatrix} = -\vec{j}\,(1+1) + \vec{k}\,(1) = -2\vec{j}+\vec{k}$$

$$\vec{A}\times(\vec{\nabla}\times\vec{A}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x-z & x+y & x+z \\ 0 & -2 & 1 \end{vmatrix} = \vec{i}\,(x+y+2x+2z) + -\vec{j}\,(x-z) + \vec{k}\,(-2x+2z)$$

$$= (3x+y+2z)\vec{i} + (-x+z)\vec{j} + (-2x+2z)\vec{k}$$

$$(\vec{A}\cdot\vec{\nabla})\vec{A} = \left((x-z)\frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y} + (x+z)\frac{\partial}{\partial z}\right)$$
$$[(x-z)\vec{i} + (x+y)\vec{j} + (x+z)\vec{k}]$$

$$= (x-z-x-z)\vec{i} + (x-z+x+y)\vec{j} + (x-z+x+z)\vec{k}$$

$$= -2z\vec{i} + (2x+y-z)\vec{j} + 2x\vec{k}$$

$$\vec{C}(x,y,z) = (3x+y)\vec{i} + (x+y)\vec{j} + 2z\vec{k}$$

$$\vec{C}(1,2,3) = 5\vec{i} + 3\vec{j} + 6\vec{k} = (5,3,6).$$

3.c (10 points) Use your response to part **3.a** of this problem to compute $\mathbf{C}(1,2,3)$ for the choices of $\mathbf{A}$ given in Equations (2) of part **3.b.**

$$|\vec{A}|^2 = (x-z)^2 + (x+y)^2 + (x+z)^2 = 2(x^2+z^2) + x^2 + y^2 + 2xy$$

$$= 3x^2 + y^2 + 2z^2 + 2xy$$

$$\vec{C} = \frac{1}{2}\vec{\nabla}|A|^2 = \frac{1}{2}\left[(6x+2y)\vec{i} + (2y+2x)\vec{j} + 4z\vec{k}\right]$$

$$= (3x+y)\vec{i} + (x+y)\vec{j} + 2z\vec{k}$$

$$\vec{C}(1,2,3) = 5\vec{i} + 3\vec{j} + 6\vec{k} = (5,3,6).$$

Problem 4 (10 points) Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F}(x, y, z) := 2xz\mathbf{i} + 2yz^2\mathbf{j} + (x^2 + 2y^2z - 1)\mathbf{k},$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors along the x-, y-, and z-axes in a Cartesian coordinate system. Find a scalar function $\phi : \mathbb{R}^3 \to \mathbb{R}$ satisfying $\mathbf{F} = \nabla\phi$.

$$F = \nabla\phi$$

$$\Rightarrow \begin{cases} \dfrac{\partial\phi}{\partial x} = 2xz & \text{(i)} \\[2mm] \dfrac{\partial\phi}{\partial y} = 2yz^2 & \text{(ii)} \\[2mm] \dfrac{\partial\phi}{\partial z} = x^2 + 2y^2z - 1 & \text{(iii)} \end{cases}$$

$\underline{\text{(i)} \Rightarrow \phi = x^2z + h(y,z)}$, where $h(y,z) : \mathbb{R}^2 \to \mathbb{R}^3$

By (ii) and (iii)

$$\frac{\partial h}{\partial y} = 2yz^2 \quad , \quad \frac{\partial h}{\partial z} = 2y^2z - 1$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

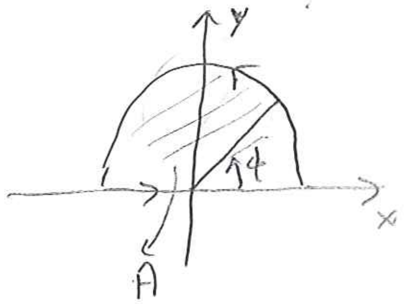
$\underline{h(y,z) = y^2z^2 + g(z)} \Rightarrow \cancel{2y^2z} + g'(z) = \cancel{2y^2z} - 1$, $g(z) : \mathbb{R} \to \mathbb{R}$

$\underline{g(z) = -z + c}$, $c \in \mathbb{R}$

$$\begin{aligned} \phi &= x^2z + h(y,z) \\ &= x^2z + y^2z^2 + g(z) \\ &= x^2z + y^2z^2 - z + c \quad , \quad c \in \mathbb{R} \end{aligned}$$

Problem ~~4~~ 5. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the force field defined by $\mathbf{F}(x,y) = y^2\mathbf{i} + x^2\mathbf{j}$ where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along the x- and y-axes.

4.a (15 points) Use Green's Theorem in plane to compute the work done by $\mathbf{F}$ along the bondy of half disc ~~semicircle~~ defined by $y \leq \sqrt{1-x^2}$ as it is traversed counterclockwise.



$$\oint \vec{F}\cdot d\vec{x} = \iint_A \left(\frac{\partial x^2}{\partial x} - \frac{\partial y^2}{\partial y}\right) dx\,dy$$

$$= 2\iint_A (x-y)\,dx\,dy$$

$(x = r\cos\phi, \quad y = r\sin\phi)$

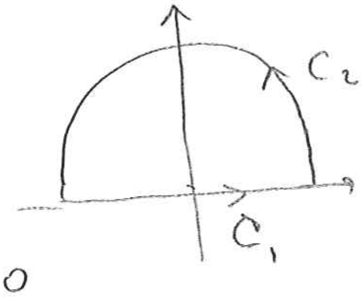
$$= 2\int_0^\pi\int_0^1 (\cos\phi - \sin\phi)\, r^2\, dr\, d\phi$$

$$= 2\left(\frac{r^3}{3}\Big|_0^1\right)\left(\sin\phi + \cos\phi\right)\Big|_0^\pi$$

$$= \frac{2}{3}(\cos\pi - \cos 0)$$

$$= -\frac{4}{3}$$

~~4~~ 5.b (15 points) Calculate the same quantity as in part 4.b by evaluating the line integral that gives the work, i.e., without using Green's Theorem.



$$\oint \vec{F}\cdot d\vec{x} = \int_{C_1} \vec{F}\cdot d\vec{x} + \int_{C_2} \vec{F}\cdot d\vec{x}.$$

$\Rightarrow$ on C_1: $x \in [-1,1]$, $y = 0$, $\vec{F} = x^2\vec{j}$, $dy = 0$

$\Rightarrow \int_{C_1} \vec{F}\cdot d\vec{x} = 0$

$\Rightarrow$ on C_2: $x = \cos\phi$, $y = \sin\phi$

$dx = -\sin\phi\, d\phi$, $y = \cos\phi\, d\phi$

$$\int_{C_2} \vec{F}\cdot d\vec{x} = \int_{C_2} (y^2 dx + x^2 dy) = \int_0^\pi \underbrace{(-\sin^3\phi + \cos^3\phi)}_{(\cos\phi - \sin\phi)(\underbrace{\cos^2\phi + \sin\phi\cos\phi + \sin^2\phi}_{1 + \sin\phi\cos\phi})} d\phi$$

$$= \int_0^\pi (\cos\phi - \sin\phi + \sin\phi\cos^2\phi - \sin^2\phi\cos\phi)\,d\phi$$

$$= \left(\sin\phi + \cos\phi - \frac{\cos^3\phi}{3} - \frac{\sin^3\phi}{3}\right)\Big|_0^\pi = -2\left(1 - \frac{1}{3}\right) = -\frac{4}{3} \checkmark$$