Solutions

Math 303: Midterm Exam 2

Fall 2008

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have One hour and 45 minutes (105 minutes).
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the exam within the first 10 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)
- (Optional) Grade your own work out of 100. Record your estimated grade here:

Estimated Grade:	

If your expected grade and actual grade will turn out to differ by 9 points or less, you will be given the highest of the two.

To be filled by the grader:

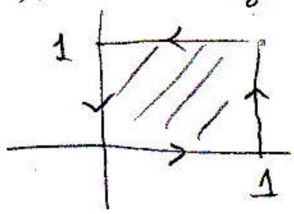
Actual Grade:	
Adjusted Grade:	

Problem 1. State and prove Green's theorem in plane for a vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ defined on the region: $D := \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ 0 \le y \le 1.\}$. (15 points)

Suppose $\vec{F}$ is differentiable in D: Then

$$\oint_{\partial D} \vec{F} \cdot dr = \iint_{D} \left(\frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} F_1 \right) dx\, dy$$

where $\vec{F} = F_1 \hat{i} + F_2 \hat{j}$ and the line integral is evaluated counterclockwise along the boundary ∂D of D.



Proof:

$$LHS := \oint_{\partial D} \vec{F} \cdot d\vec{r} = \oint_{\partial D} F_1\, dx + F_2\, dy$$

$$= \int_0^1 F_1(x,0)\, dx + \int_0^1 F_2(1,y)\, dy + \int_1^0 F_1(x,1)\, dx + \int_1^0 F_2(0,y)\, dy$$

$$= \int_0^1 [F_1(x,0) - F_1(x,1)]\, dx + \int_0^1 [F_2(1,y) - F_2(0,y)]\, dy$$

$$RHS := \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy = \int_0^1 \Bigg(\underbrace{\int_0^1 \frac{\partial F_2}{\partial x}\, dx}_{F_2(1,y) - F_2(0,y)} \Bigg) dy - \int_0^1 \Bigg(\underbrace{\int_0^1 \frac{\partial F_1}{\partial y}\, dy}_{F_1(x,1) - F_1(x,0)} \Bigg) dx$$

$$= \int_0^1 [F_2(1,y) - F_2(0,y)]\, dy + \int_0^1 [F_1(x,0) - F_1(x,1)]\, dx$$

$= LHS$ ✓ ∎

Problem 2. Find $g(x)$ such that $e^{-|x|}$ is a solution of $y'' + y = g(x)$. (10 points)

$$y = e^{-|x|}$$

$$y' = -e^{-|x|} \frac{d}{dx}|x|$$

$$|x| = x\,\theta(x) - x\,\theta(-x) = x[\theta(x) - \theta(-x)]$$

$$\frac{d}{dx}|x| = \theta(x) - \theta(-x) + \underbrace{x \underbrace{[\delta(x) - (-\delta(-x))]}_{2\delta(x)}}_{0}$$

$$= \theta(x) - \theta(-x)$$

$$\Rightarrow \quad y' = -e^{-|x|}[\theta(x) - \theta(-x)]$$

$$y'' = -\underbrace{\frac{d}{dx}(e^{-|x|})}_{y' = -e^{-|x|}[\theta(x) - \theta(-x)]}[\theta(x) - \theta(-x)] - e^{-|x|}\underbrace{\frac{d}{dx}[\theta(x) - \theta(-x)]}_{2\delta(x)}$$

$$= e^{-|x|}\underbrace{[\theta(x) - \theta(-x)]^2}_{1} - 2\underbrace{e^{-|x|}\delta(x)}_{\delta(x)}$$

$$= e^{-|x|} - 2\delta(x)$$

$$\Rightarrow \quad y'' + y' = 2(e^{-|x|} - \delta(x))$$

$$\Rightarrow \quad \boxed{g(x) = 2(e^{-|x|} - \delta(x))}$$

Problem 3. Find the real Fourier series for the following function. (15 points)

$$f(x) = \begin{cases} \sin x & \text{for } -\pi \le x < 0, \\ \cos x & \text{for } 0 \le x < \pi, \end{cases} \qquad f(x+2\pi) = f(x).$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$a_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} \sin x \cos(nx)\,dx + \int_0^{\pi} \cos x \cos(nx)\,dx\right)$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)], \qquad \sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)],$$

$$\Rightarrow a_n = \frac{1}{2\pi}\left(\int_{-\pi}^{0}(\sin[(1+n)x] + \sin[(1-n)x])\,dx + \int_0^{\pi}(\cos[(1-n)x] + \cos[(1+n)x])\,dx\right)$$

For $n \neq 1$: $$a_n = \frac{1}{2\pi}\left\{\left[-\frac{\cos[(n+1)x]}{n+1} - \frac{\cos[(-n+1)x]}{-n+1}\right]_{-\pi}^{0} + \left[\frac{\sin[(1-n)x]}{1-n} + \frac{\sin[(1+n)x]}{1+n}\right]_0^{\pi}\right\}$$

$$\Rightarrow a_n = \frac{1}{2\pi}\left\{\frac{-1+\cos[(n+1)\pi]}{n+1} + \frac{-1+\cos[(-n+1)\pi]}{-n+1}\right\} = -\frac{1}{2\pi}\left\{\frac{1+(-1)^n}{n+1} + \frac{1+(-1)^n}{-n+1}\right\}$$

$$= -\frac{1+(-1)^n}{2\pi}\left(\frac{-n+1+n+1}{-n^2+1}\right) = \boxed{\frac{1+(-1)^n}{\pi(n^2-1)} = a_n \quad n \neq 1}$$

For $n = 1$: $$a_1 = \frac{1}{2\pi}\left(\int_{-\pi}^{0}\sin(2x)\,dx + \int_0^{\pi}(1+\cos 2x)\,dx\right)$$

$$\Rightarrow a_1 = \frac{1}{2\pi}\left[-\frac{\cos 2x}{2}\Big|_{-\pi}^{0} + \left(x + \frac{\sin(2x)}{2}\right)\Big|_0^{\pi}\right] = \frac{1}{2\pi}(0+\pi) = \boxed{\frac{1}{2} = a_1}$$

$$b_n = \frac{1}{\pi}\left(\int_{-\pi}^{0}\sin x \sin(nx)\,dx + \int_0^{\pi}\cos x \sin(nx)\,dx\right)$$

$$= \frac{1}{2\pi}\left(\int_{-\pi}^{0}[\cos[(1-n)x] - \cos[(1+n)x]]\,dx + \int_0^{\pi}(\sin[(n+1)x] + \sin[(n-1)x])\,dx\right)$$

For $n \neq 1$: $$b_n = \frac{1}{2\pi}\left\{\left(\frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n}\right)\Big|_{-\pi}^{0} + \left(-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right)\Big|_0^{\pi}\right\}$$

$$\Rightarrow b_n = \frac{1}{2\pi}\left[-\frac{-(-1)^n-1}{n+1} - \frac{-(-1)^n-1}{n-1}\right] = \frac{1+(-1)^n}{2\pi}\left(\frac{1}{n+1} + \frac{1}{n-1}\right) = \frac{1+(-1)^n}{2\pi}\left(\frac{n-1+n+1}{n^2-1}\right)$$

$$\Rightarrow \boxed{b_n = \frac{[1+(-1)^n]n}{\pi(n^2-1)}, \quad n \neq 1}$$

for $n = 1$: $$b_1 = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(1-\cos 2x)\,dx + \int_0^{\pi}\sin(2x)\,dx\right]$$

$$\Rightarrow b_1 = \frac{1}{2\pi}\left[\left(x - \frac{\sin 2x}{2}\right)\Big|_{-\pi}^{0} + \left(-\frac{\cos 2x}{2}\right)\Big|_0^{\pi}\right] = \boxed{\frac{1}{2} = b_1}$$

Problem 4. Let $f(x)$ denote the inverse Fourier transform of $e^{-|k|}$. Find the Fourier transform of $(x^2+1)e^{-x}f(x)$. (15 points)

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} e^{-|k|} dk = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{(ix+1)k} dk + \int_{0}^{\infty} e^{(ix-1)k} dk \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(ix+1)k}}{ix+1} \Big|_{-\infty}^{0} + \frac{e^{(ix-1)k}}{ix-1} \Big|_{0}^{\infty} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{ix+1} - \frac{1}{ix-1} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{ix-1-ix+1}{-x^2-1} \right)$$

$$= \frac{2}{\sqrt{2\pi}(x^2+1)}$$

$$\mathcal{F}\left\{(x^2+1)e^{-|x|}f(x)\right\} = \mathcal{F}\left\{\frac{2e^{-|x|}}{\sqrt{2\pi}}\right\}$$

$$= \frac{2}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-ikx} \frac{e^{-|x|}}{\sqrt{2\pi}} dx}_{I(k) = f(-k) = \frac{2}{\sqrt{2\pi}(k^2+1)}}$$

$$= \frac{2}{\pi(k^2+1)}$$

Problem 5. A particle moves in the x-y plane in such a way that its speed is given by its distance from the origin, i.e., $r := \sqrt{x^2+y^2}$. Determine the path the particle should take to go from the $(1,0)$ to $(0,1)$ such that the travel time is minimized. (25 points)
Hint: Use polar coordinates (r,ϕ) where the line element $d\ell$ satisfies $d\ell^2 = dr^2 + r^2 d\phi^2$. Note that the speed of the particle is defined to be $d\ell/dt$.

$$\frac{d\ell}{dt} = r \quad \Rightarrow \quad dt = \frac{d\ell}{r} \quad \Rightarrow \quad t = \int \frac{d\ell}{r}$$

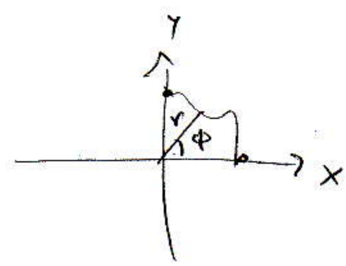
$$d\ell = \sqrt{dr^2 + r^2 d\phi^2} \qquad \text{choose} \quad \phi = \phi(r) \quad \Rightarrow \quad d\ell = \sqrt{1 + r^2 \varphi'^2}\, dr$$

$$\Rightarrow \quad t = \int \frac{\sqrt{1+r^2\varphi'^2}}{r}\, dr = \int \underbrace{\sqrt{\frac{1}{r^2} + \varphi'^2}}_{F}\, dr$$

$$\frac{\partial F}{\partial \varphi} = 0 \quad \Rightarrow \quad \frac{\partial F}{\partial \varphi'} = \underbrace{c}_{\text{const}}$$

$$\Rightarrow \quad \frac{\varphi'}{\sqrt{\frac{1}{r^2} + \varphi'^2}} = c \quad \Rightarrow \quad \frac{\varphi'^2}{\frac{1}{r^2} + \varphi'^2} = c^2 \quad \Rightarrow \quad \frac{\varphi'^2}{\frac{1}{r^2}} = \frac{c^2}{1-c^2}$$

$$\Rightarrow \quad \varphi' = \underbrace{\frac{\pm|c|}{\sqrt{1-c^2}}}_{= k} \frac{1}{r} \quad \Rightarrow \quad d\varphi = \frac{k\, dr}{r}$$



$$\Rightarrow \quad \boxed{\varphi = k \ln r + \varphi_0} \qquad \varphi_0, k \in \mathbb{R}$$

Bounds conditions: $\begin{cases} \text{Initial point } (0,1) \Rightarrow r = 1, \varphi = 0 \\ \text{final } \text{"} \ (1,0) \Rightarrow r = 1, \varphi = \frac{\pi}{2} \end{cases}$

$$\Rightarrow \begin{cases} 0 = k \underbrace{\ln 1}_{0} + \varphi_0 \quad \Rightarrow \quad \boxed{\varphi_0 = 0} \\ \frac{\pi}{2} = k \underbrace{\ln 1}_{0} \quad \Rightarrow \quad ※ \end{cases}$$

This problem does not have a solution.

Problem 6. Let $v(x, y) := y^3 - 3x^2y + y$.

a) Show that v is a solution of the Laplace equation. (5 points)

$v_x = -6xy, \quad v_{xx} = -6y$

$v_y = 3y^2 - 3x^2 + 1 \Rightarrow v_{yy} = 6y$

$\nabla^2 v = v_{xx} + v_{yy} = -6y + 6y = 0 \Rightarrow \boxed{\nabla^2 v = 0}$ ✓

b) Find an analytic function $f : \mathbb{C} \to \mathbb{C}$ such that $v(x, y)$ is the imaginary part of $f(x + iy)$ and $f(0) = 1$. Give an explicit formula for $f(z)$. (15 points)

$u_x = v_y = 3y^2 - 3x^2 \qquad u = \int (3y^2 - 3x^2 + 1)\,dx + g(y)$

$\Rightarrow u = 3y^2x - x^3 + x + g(y)$

$u_y = -v_x \Rightarrow 6yx + g'(y) = -(-6xy)$

$\Rightarrow g'(y) = 0 \Rightarrow g(y) = c$

const.

$\Rightarrow \boxed{u = 3xy^2 - x^3 + x + c}$

$\Rightarrow f(x+iy) = 3xy^2 - x^3 + x + c + i(y^3 - 3x^2y + y)$

$\Rightarrow f(x) = -x^3 + x + c \Rightarrow f(z) = -z^3 + z + c$

$f(0) = 1 \Rightarrow c = 1$

$\Rightarrow \boxed{f(z) = -z^3 + z + 1}$