Solutions

Math 303: Midterm Exam 1

Fall 2008

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have One hour and 45 minutes (105 minutes).
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the exam within the first 10 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)
- (Optional) Grade your own work out of 100. Record your estimated grade here:

Estimated Grade:	

If your expected grade and actual grade will turn out to differ by 9 points or less, you will be given the highest of the two.

To be filled by the grader:

Actual Grade:	
Adjusted Grade:	

Problem 1. Find the stationary points of $f(x,y) = (x^2+2)^2 + (y^2+2)^2 - 6x^2y^2 - 8$ and apply the second derivative test to determine if they are minimum, maximum, or saddle points. (20 points)

$$f_x = 2(x^2+2)(2x) - 12xy^2 = 4x(x^2-3y^2+2) = 0 \quad (1)$$

$$f_y = 2(y^2+2)(2y) - 12yx^2 = 4y(y^2-3x^2+2) = 0 \quad (2)$$

$$f_{xx} = 4(x^2-3y^2+2+2x^2) = 4(3x^2-3y^2+2)$$

$$f_{xy} = -24xy$$

$$f_{yy} = 4(y^2-3x^2+2+2y^2) = 4(3y^2-3x^2+2)$$

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① If $x=0 \overset{(2)}{\Rightarrow} y(y^2+2)=0 \Rightarrow y=0 \Rightarrow \boxed{P_1=(0,0)}$

② If $x \neq 0 \Rightarrow x^2-3y^2+2=0 \Rightarrow 3y^2 = x^2+2>0 \Rightarrow y\neq 0$

$\overset{(2)}{\Rightarrow} y^2-3x^2+2=0$ subtract $x^2-3y^2-y^2+3x^2=0$

$4(x^2-y^2)=0 \Rightarrow y=\pm x$

$-2x^2+2=0 \Rightarrow x=\pm 1 \Rightarrow y=\pm 1$

$\Rightarrow \boxed{P_2=(-1,-1),\ P_3=(-1,1),\ P_4=(1,-1),\ P_5=(1,1)}$

$\mathbb{H} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} \Rightarrow P_1=(0,0) \Rightarrow \mathbb{H}(P_1) = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} \Rightarrow \boxed{P_1 \text{ is a min.}}$

$P_2=(-1,-1) \Rightarrow \mathbb{H}(P_2) = \begin{pmatrix} 8 & -24 \\ -24 & 8 \end{pmatrix}$ $tr(\mathbb{H})>0$, $det(\mathbb{H})<0 \Downarrow$ $\boxed{P_2 \text{ is a saddle point.}}$

$P_3=(-1,1) \Rightarrow \mathbb{H}(P_3) = \begin{pmatrix} 8 & 24 \\ 24 & 8 \end{pmatrix} \Rightarrow det\ \mathbb{H}<0 \Rightarrow \boxed{P_3 \text{ is a saddle point.}}$

$P_4=(1,-1) \Rightarrow \mathbb{H}(P_3) = \begin{pmatrix} 8 & 24 \\ 24 & 8 \end{pmatrix} \Rightarrow$ " $\Rightarrow \boxed{P_4 \text{ is a saddle point.}}$

$P_5=(1,1) \Rightarrow \mathbb{H}(P_5) = \begin{pmatrix} 8 & -24 \\ -24 & 8 \end{pmatrix} \Rightarrow det\ \mathbb{H}<0 \Rightarrow \boxed{P_5 \text{ is a saddle point}}$

Problem 2. Find the points on the paraboloid $z = (x-1)^2 + (y-2)^2 + \frac{3}{2}$ that are closest to the point $(1,2,3)$. (20 points)

$$l^2 = (x-1)^2 + (y-2)^2 + (z-3)^2$$

$$\phi := (x-1)^2 + (y-2)^2 - z + \frac{3}{2} = 0$$

$$F = l^2 + \lambda\phi = (x-1)^2 + (y-2)^2 + (z-3)^2 + \lambda\left[(x-1)^2 + (y-2)^2 - z + \frac{3}{2}\right]$$

$$F_x = 2(x-1) + \lambda[2(x-1)] = 2(x-1)(\lambda+1) = 0 \quad (1)$$

$$F_y = 2(y-2) + \lambda[2(y-2)] = 2(y-2)(\lambda+1) = 0 \quad (2)$$

$$F_z = 2(z-3) - \lambda = 0 \quad (3)$$

$$F_\lambda = (x-1)^2 + (y-2)^2 - z + \frac{3}{2} = 0 \quad (4)$$

Either: (I) $\lambda = -1 \overset{(3)}{\Rightarrow} z - 3 = -\frac{1}{2} \Rightarrow \boxed{z = \frac{5}{2}}$

$\overset{4}{\Rightarrow} (x-1)^2 + (y-2)^2 - \frac{5}{2} + \frac{3}{2} = 0 \Rightarrow \boxed{(x-1)^2 + (y-2)^2 = 1}$

or: (II) $\lambda \neq -1 \Rightarrow \boxed{x = 1,\ y = 2} \overset{(4)}{\Rightarrow} \boxed{z = \frac{3}{2}}$

So we find the point on the circle $(x-1)^2 + (y-2)^2 = 1$ and $z = \frac{5}{2}$ with distance

$$l = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2} = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{4} \approx \frac{2.2}{4} = 0.55$$

and the point $p = (1, 2, \frac{3}{2})$ with distance

$$l = \sqrt{(\frac{3}{2} - 3)^2} = \frac{3}{2} > \frac{\sqrt{5}}{2}$$

So the closest points are those on the circle $(x-1)^2 + (y-2)^2 = 1$ & $z = \frac{5}{2}$.

Problem 3.

a) Let ϵ_{ijk} denote the Levi Civita tensor. Show that for every 3×3 matrix M with entries M_{ij} we have $\sum_{i,j,k=1}^{3} \epsilon_{ijk} M_{ij} M_{ik} = 0.$ (10 points)

$$\text{LHS} = \sum_{i,j,k=1}^{3} \epsilon_{ijk} M_{ij} M_{ik} = \underbrace{\sum_{i,j,k=1}^{3} \epsilon_{ikj} M_{ik} M_{ij}}_{j \leftrightarrow k \text{ (relabel)}} = \sum_{i,j,k=1}^{3} \epsilon_{ikj} M_{ij} M_{ik}$$

$$= - \sum_{i,j,k=1}^{3} \epsilon_{ijk} M_{ij} M_{ik} = -\text{LHS}$$

$$\Rightarrow \quad 2\,\text{LHS} = 0 \quad \Rightarrow \quad \text{LHS} = 0 = \text{RHS}. \quad \blacksquare$$

b) Let $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ and $\phi : \mathbb{R}^3 \to \mathbb{R}$ be twice differentiable functions. Use properties of the Levi Civita symbol to show that $\nabla \cdot (\nabla f \times \mathbf{A}) = -\nabla f \cdot (\nabla \times \mathbf{A})$. (10 points)

$$\vec{\nabla} \cdot (\vec{\nabla} f \times \vec{A}) = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \sum_{j,k=1}^{3} \left[\epsilon_{ijk} \frac{\partial f}{\partial x_j} A_k \right]$$

$$= \sum_{i,j,k=1}^{3} \epsilon_{ijk} \left[\frac{\partial^2 f}{\partial x_i \partial x_j} A_k + \frac{\partial f}{\partial x_j} \frac{\partial A_k}{\partial x_i} \right]$$

$$= \sum_{k=1}^{3} \left(\sum_{i,j=1}^{3} \left(\epsilon_{ijk} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} f \right) \right) A_k + \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} \left(\sum_{i,k=1}^{3} \epsilon_{ijk} \frac{\partial}{\partial x_i} A_k \right)$$

$$= \sum_{k=1}^{3} \Big(\underbrace{\sum_{i,j=1}^{3} \epsilon_{kij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} f}_{(\vec{\nabla} \times \vec{\nabla} f)_k = 0} \Big) A_k - \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} \Big(\underbrace{\sum_{i,k=1}^{3} \epsilon_{jik} \frac{\partial}{\partial x_i} A_k}_{(\vec{\nabla} \times \vec{A})_j} \Big)$$

$$= - \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} (\vec{\nabla} \times \vec{A})_j$$

$$= -\vec{\nabla} f \cdot (\vec{\nabla} \times \vec{A})$$

Problem 4. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the force field given by

$$\mathbf{F} = [3x^2y + f(y)]\mathbf{i} + [-4y^3x + g(x)]\mathbf{j} + \mathbf{k}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable functions.

a) Find the general form of f and g so that $\mathbf{F}$ is a conservative force. (10 points)

$$\vec{0} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y + f(y) & -4y^3x + g(x) & 1 \end{vmatrix} = [-4y^3 + g'(x) - 3x^2 - f'(y)]\hat{k}$$

$$\Rightarrow \quad g'(x) - 3x^2 - [f'(y) + 4y^3] = 0 \quad \Rightarrow \quad \begin{cases} g'(x) - 3x^2 = c_1 \\ f'(y) + 4y^3 = c_1 \end{cases} \quad \text{for some } c_1 \in \mathbb{R}$$

$$\Rightarrow \begin{cases} g(x) = x^3 + c_1 x + c_2 \\ f(y) = -y^4 + c_1 y + c_3 \end{cases} \quad \text{for some } c_2, c_3 \in \mathbb{R}$$

b) For the case that $f(1) = g(1) = 0$ and $g'(0) = 1$ compute the work done by $\mathbf{F}$ on a particle that moves from the point $(1, 0, 0)$ to the point $(0, 1, 0)$ along the semicircle $y = \sqrt{1 - x^2}$ in the x-y-plane counterclockwise. (20 points)

$$f(1) = 0 \Rightarrow \quad -1 + c_1 + c_3 = 0 \Rightarrow \quad c_3 = 0$$

$$g(1) = 0 \Rightarrow \quad 1 + c_1 + c_2 = 0 \Rightarrow \quad c_2 = -2$$

$$g'(0) = 1 \Rightarrow \quad c_1 = 1$$

so $g(x) = x^3 + x - 2, \quad f(y) = -y^4 + y$

$$\vec{F} = -\vec{\nabla} V \Rightarrow \quad V_x = -(3x^2y - y^4 + y)$$

$$V_y = -(-4y^3x + x^3 + x - 2)$$

$$V_z = -1 \quad \Rightarrow \quad \boxed{V = -z + h(x,y)}$$

$$\Rightarrow \quad -(3x^2y - y^4 + y) = V_x = h_x \Rightarrow \quad h = -(x^3y - xy^4 + xy) + e(y)$$

$$\Rightarrow \quad V = -(x^3y - xy^4 + xy + z) + e(y)$$

$$\Rightarrow \quad -(-4y^3x + x^3 + x - 2) = V_y = -(x^3 - 4xy^3 + x) + e'(y)$$

$$\Rightarrow \quad e'(y) = 2 \Rightarrow \quad e(y) = 2y + k \quad \text{for some } k \in \mathbb{R}$$

$$\Rightarrow \quad \boxed{V = -(x^3y - xy^4 + xy + z - 2y) + k}$$

$$V(1,0,0) = k$$

$$V(0,1,0) = -(-2) + k = k + 2$$

$$W = V(1,0,0) - V(0,1,0) = k - (k+2) = -2 \quad \Rightarrow \quad \boxed{W = -2}$$

Problem 5.

Let r be a positive real number and z be a nonzero complex number that satisfies

$$(z-r)^2 = r^2 e^z.$$

Show that z must have a positive real part. (10 points)

Hint: Compute the modulus of both sides of the above equation.

$$|(z-r)^2| = |r^2 e^z|$$

$$\Rightarrow \quad |z-r|^2 = r^2 |e^z|$$

$$z = Re(z) + i\, Im(z)$$

$$\Rightarrow \quad e^z = e^{Re(z)} e^{i Im(z)}$$

$$\Rightarrow \quad |e^z| = |e^{Re(z)}| \underbrace{|e^{i Im(z)}|}_{1} = e^{Re(z)}$$

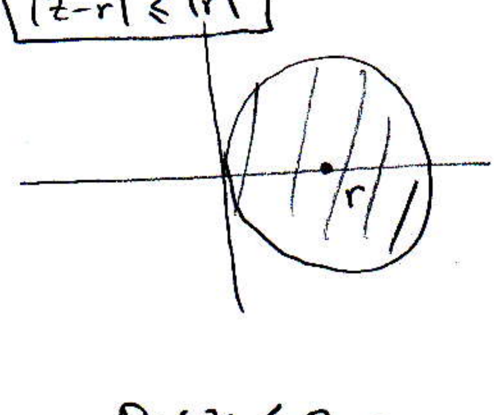
$$\Rightarrow \quad |z-r|^2 = r^2 e^{Re(z)}$$

Suppose $Re(z) \leq 0 \Rightarrow e^{Re(z)} \leq 1 \Rightarrow |z-r|^2 \leq r^2$

$\Rightarrow \boxed{|z-r| \leq |r|}$

This inequality implies that z belongs to the disc

$$D = \{ w \in \mathbb{C} \mid |w-r| \leq r \}$$



$\Rightarrow \quad z \in D \Rightarrow Re(z) \geq 0$

This together with the assumption $Re(z) \leq 0$ imply $Re(z) = 0$. But the only $z \in D$ with $Re(z) = 0$ is $z = 0$. This contradicts the fact that $z \neq 0$. Therefore the assumption $Re(z) \leq 0$ is false $\Rightarrow Re(z) > 0$.