Math 303: Quiz # 4

Fall 2008

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 50 minutes.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)

****

1. Let $\delta'(x)$ denote the derivative of the Dirac delta function. Show that $\delta'(-x) = -\delta'(x)$. (20 points)

Hint: Let $\alpha(x) := \delta'(-x)$ and use one of the conditions for equality of two generalized functions to show that $\alpha(x) = -\delta'(x)$. (20 points)

Let f be a differentiable test function such that $f(x) \to 0$ as $x \to \pm\infty$. Then

$$\int_{-\infty}^{\infty} f(y)\, \alpha(y-x)\, dy = \int_{-\infty}^{\infty} f(y)\, \delta'(-y+x)\, dy \qquad \text{let } \begin{array}{l} u := -y \\ dy = -du \end{array}$$

$$= \int_{\infty}^{-\infty} -f(-u)\, \delta'(u+x)\, du$$

$$= \int_{-\infty}^{\infty} f(-u)\, \delta'(u-(-x))\, du$$

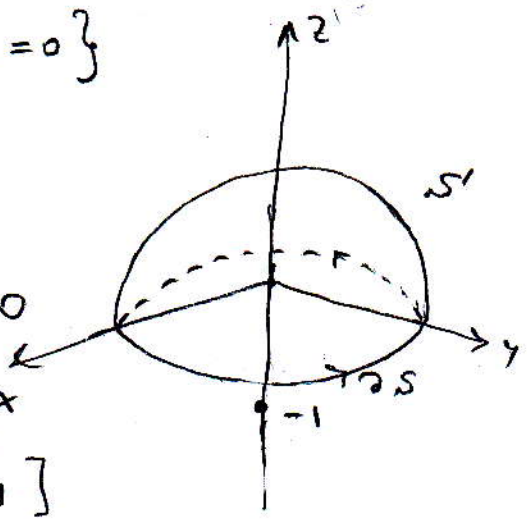
$$= -\frac{d}{du} f(-u)\Big|_{u=-x} = f'(-u)\Big|_{u=-x} = f'(x)$$

$$= -\int_{-\infty}^{\infty} f(y)\, \delta'(y-x)\, dy = \int_{-\infty}^{\infty} f(y)\, [-\delta'(y-x)]\, dy$$

$$\Rightarrow \alpha(y-x) = -\delta'(y-x) \Rightarrow \alpha(x) = -\delta'(x). \quad \blacksquare$$

2. Verify the statement of Stokes' theorem for the the surface S and vector field $\mathbf{F}$ defined by $S := \{(x,y,z) \in \mathbb{R}^3 | (z+1)^2 + x^2 + y^2 = 5,\ z \geq 0\}$, $\mathbf{F} = (-x+y+z)\mathbf{i} + (x-y+z)\mathbf{j} + (x+y-z)\mathbf{k}$, by calculating the relevant line and surface integrals. (30 points)

$$\partial S = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 4,\ z = 0 \}$$



$$I_1 := \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

on ∂S: $x = 2\cos\theta$, $y = 2\sin\theta$, $z = 0$

$dx = -2\sin\theta\, d\theta$, $dy = 2\cos\theta\, d\theta$

$$I_1 = \int_{\partial S} \left[(-x + y + \overset{\to 0}{z}) dx + (x - y + \overset{\to 0}{z}) dy \right]$$

$$= \int_0^{2\pi} [(-2\cos\theta + 2\sin\theta)(-2\sin\theta) + (2\cos\theta - 2\sin\theta)(2\cos\theta)] d\theta$$

$$= 4\int_0^{2\pi} (\sin\theta\cos\theta \underbrace{- \sin^2\theta + \cos^2\theta}_{\cos 2\theta} - \sin\theta\cos\theta)\, d\theta$$

$$= 4\int_0^{2\pi} \cos 2\theta\, d\theta = 4 \frac{\sin 2\theta}{2} \Big|_0^{2\pi} = 0 \Rightarrow \boxed{I_1 = 0}$$

$$I_2 := \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, d\sigma$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x+y+z & x-y+z & x+y-z \end{vmatrix} = \hat{i}(1-1) - \hat{j}(1-1) + \hat{k}(1-1)$$

$$= \vec{0}$$

$$\Rightarrow \boxed{I_2 = 0} \quad \Rightarrow \quad \boxed{I_1 = I_2}\ ✓$$

3. Let $\Theta : \mathbb{R} \to \mathbb{R}$ be defined by $\Theta := \begin{cases} 0 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad x > 0 \end{cases}$.

3.a) Find $f, g : \mathbb{R} \to \mathbb{R}$ such that $|x| = \Theta(x)f(x) + \Theta(-x)g(x)$. (10 points)

For $x > 0$: $\quad x = |x| = f(x) \Rightarrow \quad f(x) = x$

For $x < 0$: $\quad -x = |x| = g(x) \Rightarrow \quad g(x) = -x$

3.b) Calculate the first two derivatives of $|x|$. (15 points)

$$|x| = x\,\theta(x) + (-x)\,\theta(-x) = x\left[\theta(x) - \theta(-x)\right]$$

$$\frac{d}{dx}|x| = \left[\theta(x) - \theta(-x)\right] + x\frac{d}{dx}\left[\theta(x) - \theta(-x)\right]$$

$$= \theta(x) - \theta(-x) + x\left[\delta(x) - (-\delta(-x))\right]$$

$$= \theta(x) - \theta(-x) + \underbrace{2x\delta(x)}_{= 0 \cdot \delta(x) = 0.}$$

$$= \theta(x) - \theta(-x)$$

$$\frac{d^2}{dx^2}|x| = \frac{d}{dx}\left[\theta(x) - \theta(-x)\right]$$

$$= \delta(x) - (-\delta(-x))$$

$$= 2\,\delta(x).$$

3.c) Calculate $e^{|x-1|}\frac{d^2}{dx^2}e^{-|x-1|}$. (25 points)

$$\frac{d}{dx} e^{-|x-1|} = -\frac{d}{dx}(|x-1|)\, e^{-|x-1|}$$

$$= -\left[\theta(x-1)-\theta(-x+1)\right] e^{-|x-1|}$$

by (3.b)

$$= \left[-\theta(x-1)+\theta(-x+1)\right] e^{-|x-1|}$$

$$\frac{d^2}{dx^2} e^{-|x-1|} = \left[-\delta(x-1)-\delta(-x+1)\right] e^{-|x-1|} + \left[-\theta(x-1)+\theta(-x+1)\right]\frac{d}{dx} e^{-|x-1|}$$

$$= -2\delta(x-1)\, e^{-|x-1|} + \left[-\theta(x-1)+\theta(-x+1)\right]\left[-\theta(x-1)+\theta(-x+1)\right] e^{-|x-1|}$$

$$\Rightarrow e^{|x-1|}\frac{d^2}{dx^2} e^{-|x-1|} = -2\delta(x-1) + \underbrace{\left[-\theta(x-1)+\theta(-x+1)\right]^2}_{\alpha(x)}$$

$$\alpha(x) = \begin{cases} 1 & \text{for } 1>0 \\ 1 & \text{for } 1<0 \end{cases} = 1$$

↳ as a generalized fn.

$$\Rightarrow \boxed{e^{|x-1|}\frac{d^2}{dx^2} e^{-|x-1|} = -2\delta(x-1)}$$