Solutions

Math 303: Quiz # 1

Fall 2008

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 50 minutes.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)

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1. Compute $(-i)^{\frac{1}{4}}$ and express your response in polar form. (20 points)

$$-i = e^{-\frac{i\pi}{2} + 2\pi i n} \qquad n \in \mathbb{Z}$$

$$\Rightarrow (-i)^{\frac{1}{4}} = \left(e^{-\frac{i\pi}{2} + 2\pi i n}\right)^{\frac{1}{4}} = e^{-\frac{i\pi}{8} + \frac{\pi i n}{2}}$$

$$= \begin{cases} e^{-\frac{i\pi}{8}} & \text{for} \quad n=0 \\ e^{\frac{3\pi i}{8}} & \text{for} \quad n=1 \\ e^{\frac{7\pi i}{8}} & \text{for} \quad n=2 \\ e^{\frac{11\pi i}{8}} = e^{-\frac{5\pi i}{8}} & \text{for} \quad n=3 \ \& \ n=-1. \end{cases}$$

2. Let R be the locus of points z in the complex plane such that $|z|^2 \le |z^2-1|$. Find a relation involving x and y (where $z = x+iy$ and $x, y \in \mathbb{R}$) that describes R and draw R in complex plane. (20 points)

$$|z|^2 = x^2+y^2$$

$$z^2-1 = x^2-y^2+2ixy-1 = (x^2-y^2-1)+i(2xy)$$

$$|z^2-1|^2 = (x^2-y^2-1)^2+4x^2y^2$$

$$= x^4+y^4+1-2x^2y^2-2x^2+2y^2+4x^2y^2$$

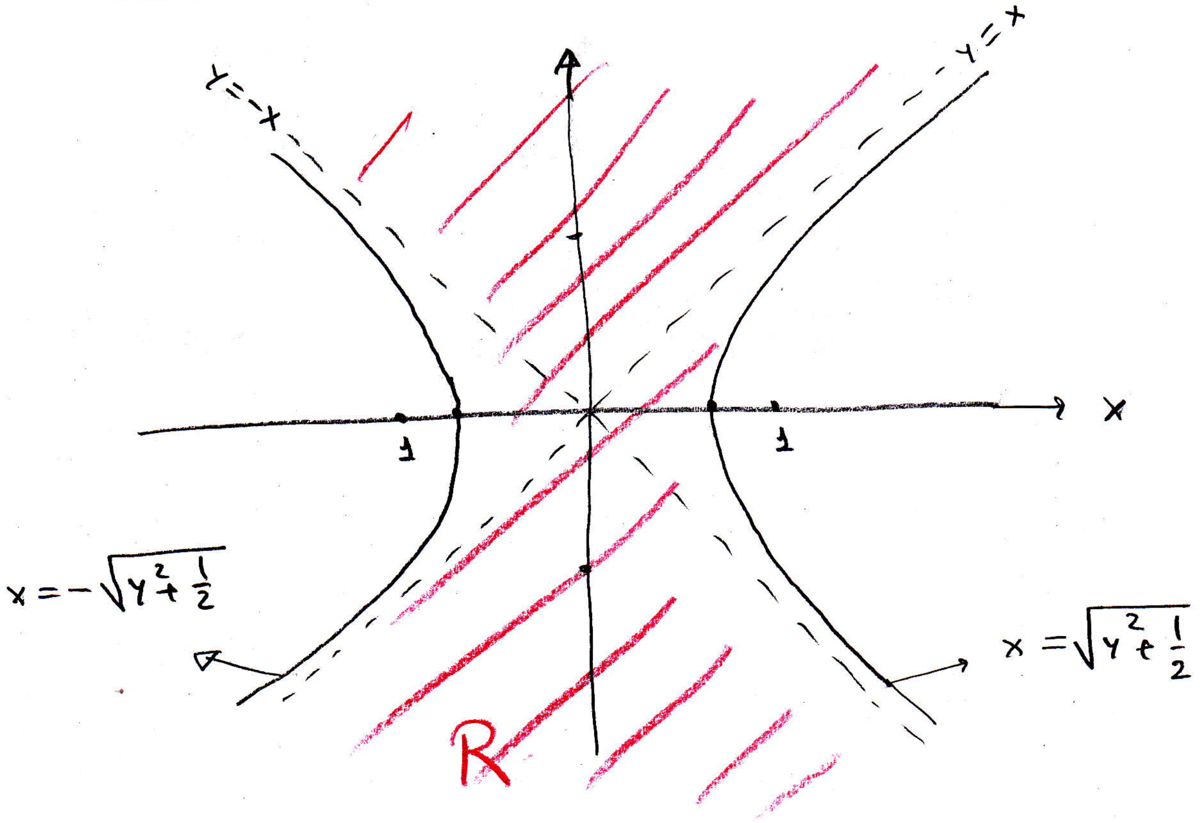
$$= \underbrace{x^4+y^4+2x^2y^2}_{|z|^4}-2x^2+2y^2+1$$

$$|z|^2 \le |z^2-1| \iff |z|^4 \le |z^2-1|^2$$

$$\Rightarrow |z|^4 \le |z|^4 - 2x^2+2y^2+1$$

$$\Rightarrow x^2 \le y^2+\frac{1}{2} \Rightarrow |x| \le \sqrt{y^2+\frac{1}{2}}$$

$$\Rightarrow \boxed{-\sqrt{y^2+\frac{1}{2}} \le x \le \sqrt{y^2+\frac{1}{2}}}$$



3. Find the real solutions of the equation: $\cosh(4x) - 4\cosh(2x) - 5 = 0$. (30 points)

$$\cosh(4x) = \frac{e^{4x} + e^{-4x}}{2} = \frac{1}{2}\Big[\underbrace{(e^{2x} + e^{-2x})^2}_{2\cosh(2x)} - 2\Big]$$

$$= 2\cosh^2(2x) - 1$$

let $y := \cosh(2x) \Rightarrow 2y^2 - 1 - 4y - 5 = 0$

$$\Rightarrow 2y^2 - 4y - 6 = 0 \Rightarrow \underbrace{y^2 - 2y - 3 = 0}_{(y-3)(y+1) = 0}$$

$$\Rightarrow y = 3 \quad \text{or} \quad y = -1$$

Because $y = \cosh(2x) = \frac{1}{2}(e^{2x} + e^{-2x}) \geq 1$ for $x \in \mathbb{R}$

we have $\underbrace{\cosh(2x) = 3}$

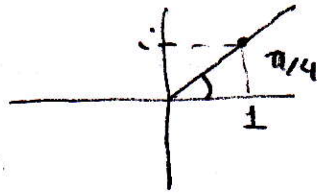
$$\frac{1}{2}(e^{2x} + e^{-2x}) = 3$$

let $u := e^{2x} \Rightarrow u + \frac{1}{u} = 6 \Rightarrow u^2 - 6u + 1 = 0$

$\Rightarrow u = 3 \pm \sqrt{9-4} = 3 \pm \sqrt{5} > 0$ for both $\pm$

$$\Rightarrow \boxed{x = \frac{1}{2}\ln(u) = \frac{1}{2}\ln(3 \pm \sqrt{5}).}$$

4. Compute $\left|(1+i)^{1-i}\right|$. (30 points):



$$(1+i)^{1-i} = e^{(1-i)\,\mathrm{Ln}(1+i)}$$

$$1+i = \sqrt{2}\, e^{i\frac{\pi}{4} + 2\pi i n} \qquad n \in \mathbb{Z}$$

$$\Rightarrow \mathrm{Ln}(1+i) = \ln(\sqrt{2}) + i\left(\frac{\pi}{4} + 2\pi n\right)$$

$$(1-i)\,\mathrm{Ln}(1+i) = (1-i)\left[\ln\sqrt{2} + i\left(\frac{\pi}{4} + 2\pi n\right)\right]$$

$$= \ln\sqrt{2} + \frac{\pi}{4} + 2\pi n + i\left(-\ln\sqrt{2} + \frac{\pi}{4} + 2\pi n\right)$$

$$\Rightarrow (1+i)^{1-i} = e^{\ln\sqrt{2} + \frac{\pi}{4} + 2\pi n + i\left(-\ln\sqrt{2} + \frac{\pi}{4} + 2\pi n\right)}$$

$$= e^{\ln\sqrt{2}}\, e^{(2n+\frac{1}{4})\pi}\, e^{i[-\ln\sqrt{2} + (2n+\frac{1}{4})\pi]}$$

$$= \sqrt{2}\, e^{(2n+\frac{1}{4})\pi}\, e^{i[-\ln\sqrt{2} + (2n+\frac{1}{4})\pi]}$$

$$\Rightarrow \left|(1+i)^{1-i}\right| = \left|\sqrt{2}\, e^{(2n+\frac{1}{4})\pi}\right| \underbrace{\left|e^{i[-\ln\sqrt{2} + (2n+\frac{1}{4})\pi]}\right|}_{1}$$

$$\Rightarrow \boxed{\left|(1+i)^{1-i}\right| = \sqrt{2}\, e^{(2n+\frac{1}{4})\pi} \quad \text{when } n \in \mathbb{Z}}$$