Math 303: Midterm Exam # 2

Fall 2006

- Write your name and Student ID number in the space provided below and sign.

Student's Name:	
ID Number:	
Signature:	

- Make sure that your exam paper consists of 5 problems (6 pages)
- You have 80 minutes.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the exam within the first 10 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)
- (Optional) Grade your own work out of 100. Record your estimated grade here:

Estimated Grade:	

If your expected grade and actual grade will turn out to differ by 9 points or less, you will be given the highest of the two.

To be filled by the grader:

Actual Grade:	
Adjusted Grade:	

Problem 1.

a) Show that $\int_{-\infty}^{\infty} e^{ixy} dy = 2\pi\delta(x)$ for all $x \in \mathbb{R}$, where $\delta(x)$ is the Dirac delta function. (10 points)

b) The integral $I(x) := \displaystyle\int_{-\infty}^{\infty} \frac{e^{ixy}}{y}\, dy$ may be viewed as the solution of the differential equation $I'(x) = 2\pi i\delta(x)$ that is an odd function $(I(-x) = -I(x))$. Use these properties to express $I(x)$ in terms of the step function:

$$\theta(x) := \begin{cases} 0 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad x > 0. \end{cases} \qquad \text{(5 points)}$$

Problem 2. Use the method of Fourier transform to obtain a particular solution of the differential equation: $y'' + y = \delta(x)$, where $\delta(x)$ is the Dirac delta function. (25 points)

Hint: You may use the following formula

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{1+\nu k}\, dk = \nu i\pi\, e^{-\nu i x}\, \mathrm{sign}(x),$$

where $\nu \in \{-1, 1\}$ and $\mathrm{sign}(x) := \begin{cases} -1 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad x > 0, \end{cases}$.

Problem 3. Determine a geodesic on the cylinder: $S := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\ \}$ that joins the points $\vec{p}_1 = (1, 0, 0)$ and $\vec{p}_2 = (0, 1, 1)$. (20 points)

Hint: Use cylindrical coordinates (r, θ, z) and express the geodesic as $z = z(\theta)$.

Problem 4. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = e^z$ for all $z \in \mathbb{C}$.

a) Determine the real and imaginary parts $u(x, y)$ and $v(x, y)$ of $f(x + iy)$ for all $x, y \in \mathbb{R}$. (10 points)

b) Prove that f is an entire function. (10 points)

Problem 5. Evaluate the following contour integrals along the circle $C := \{z \in \mathbb{C} \mid |z| = 3\ \}$ (counterclockwise).

a) $\displaystyle\oint_C \frac{\sin(\frac{\pi z}{4})}{z-2}\, dz.$ (5 points)

b) $\displaystyle\oint_C \frac{\sin(\frac{\pi z}{4})}{(z-2)^2}\, dz.$ (5 points)

c) $\displaystyle\oint_C \frac{\sin(\frac{\pi z}{4})}{(z-2)(z+4)}\, dz.$ (10 points)