Solution:

Name: Student ID: Signature:

Math 303: Quiz # 3

Fall 2004

- You have 35 minutes.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask 2 points will be deduced from your grade.

1. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |\sin x|$ satisfies $\frac{d^2}{dx^2} f(x) + f(x) = 2\delta(x)$, where δ denotes the Dirac delta function. (9 points)

$f(x) = (\sin x)\,\alpha(x)$ where $\alpha(x) = \begin{cases} -1 & \text{if } x < 0 \\ +1 & \text{if } x \geq 0 \end{cases}$

$\alpha(x) = \theta(x) - \theta(-x)$ when $\theta(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$

$\Rightarrow \quad f(x) = \sin x\,[\theta(x) - \theta(-x)]$

$f'(x) = \cos x\,[\theta(x) - \theta(-x)] + \underbrace{\sin x\,[\delta(x) + \delta(x)]}_{0}$

$\Rightarrow \quad f''(x) = -\sin x\,[\theta(x) - \theta(-x)] + \cos x\,[\delta(x) + \delta(x)]$

$= -f(x) + 2\cos(x)\,\delta(x)$

$= -f(x) + 2\,\delta(x)$

$\Rightarrow \quad \boxed{f''(x) + f(x) = 2\,\delta(x)}$

2. Let $\mathcal{H}$ be the space of square-integrable functions $f : \mathbb{R} \to \mathbb{C}$ and $\langle\cdot|\cdot\rangle$ be the L^2-inner product on $\mathcal{H}$, i.e., for all $f, g \in \mathcal{H}$, $\langle f|g\rangle = \int_{\mathbb{R}} \overline{f(x)} g(x)\,dx$. Suppose that for all $n \in \mathbb{Z}^+$, $\alpha_n, \beta_n \in \mathcal{H}$ be such that for all $f \in \mathcal{H}$ there are $a_n, b_n \in \mathbb{C}$ satisfying $f = \sum_{n=1}^{\infty} a_n\,\alpha_n$ and $f = \sum_{n=1}^{\infty} b_n\,\beta_n$. Show that if $\langle\alpha_m|\beta_n\rangle = \delta_{mn}$ for all $n, m \in \mathbb{Z}^+$, then $\sum_{n=1}^{\infty} |\alpha_n\rangle\langle\beta_n| = I$, where δ_{mn} is the Kronecker delta and I is the identity operator. (5 points)

$\forall f \in \mathcal{H}, \quad \sum_{n=1}^{\infty} |\alpha_n\rangle\langle\beta_n|f\rangle = \sum_{n=1}^{\infty} |\alpha_n\rangle\left[\langle\beta_n|\sum_{m=1}^{\infty} a_m|\alpha_m\rangle\right]$

$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\alpha_n\rangle a_m \underbrace{\langle\beta_n|\alpha_m\rangle}_{\langle\alpha_m|\beta_n\rangle = \delta_{mn}}$

$= \sum_{n=1}^{\infty} a_n|\alpha_n\rangle = |f\rangle$

$\Rightarrow \quad \sum_{n=1}^{\infty} |\alpha_n\rangle\langle\beta_n| = I.$

3) Let $\mathcal{H}$ and $\langle\cdot|\cdot\rangle$ be as in Problem 2, and $A, B : \mathcal{H} \to \mathcal{H}$ be linear operators.

3.a) Give the definition of the adjoint $A^\dagger$ of A. (1 point)

$A^+ : \mathcal{H} \to \mathcal{H}$ is the operator satisfy.

$$\langle f, A^+ g\rangle = \langle Af, g\rangle, \text{ for all } f, g \in \mathcal{H}. \quad (1)$$

3.b)] Show that if B satisfies

$$\langle f|Bg\rangle = \langle Af|g\rangle, \quad \text{for all } f, g \in \mathcal{H} \qquad (2)$$

then $B = A^\dagger$. (5 points)

$(1) \& (2) \implies \quad \langle f|Bg\rangle \overset{(3)}{=} \langle f|A^+ g\rangle, \quad \forall f, g \in \mathcal{H}$

$\Downarrow$

$\langle f|(B - A^+)g\rangle = 0, \quad \forall f, g \in \mathcal{H}$

$\Downarrow$

let $f = (B - A^+)g$ $\qquad \|(B - A^+)g\|^2 = 0, \quad \forall g \in \mathcal{H}$

$\Downarrow$

$(B - A^+)g = 0, \quad \forall g \in \mathcal{H}$

$\Downarrow$

$B - A^+ = 0$

$\Downarrow$

$B = A^+$

Alternatively $(3) \implies Bg = A^+ g \quad \forall g \in \mathcal{H}$ according to a prop. we proved in class

$\Downarrow$

$B = A^+$
