Page 393 Problem 12

we want to minimize $\int n\,ds$ where $ds = \sqrt{dx^2+dy^2}$

& $n = y^{-1}$

$$\int n\,ds = \int \frac{1}{y}\sqrt{dx^2+dy^2} =: I$$

(I) can be written as in two different ways

1) $I = \int \frac{1}{y}\sqrt{1+y'^2}\,dx$ or 2) $\int \frac{1}{y}\sqrt{1+x'^2}\,dy$

Second one is better since if we think x as the dependent variable, absence of x in the integrand makes the calculations easier.

so $F(y,x,x') = \frac{1}{y}\sqrt{1+x'^2}$

$$\frac{d}{dy}\frac{\partial F}{\partial x'} - \frac{\partial F}{\partial x} = 0$$

$$\frac{\partial F}{\partial x} = 0 \qquad \frac{\partial F}{\partial x'} = \frac{x'}{y\sqrt{1+x'^2}}$$

Euler Eq implies $\dfrac{x'}{y\sqrt{1+x'^2}} = c$ for some constant $c \neq 0$

$$\Rightarrow \frac{x'}{\sqrt{1+x'^2}} = cy \Rightarrow \frac{x'^2}{1+x'^2} = c^2y^2$$

$$\Rightarrow \quad x'^2 = c^2y^2 + x'^2c^2y^2$$

$$\Rightarrow \quad x'^2(1-c^2y^2) = c^2y^2$$

$$\Rightarrow \quad x'^2 = \frac{c^2y^2}{1-c^2y^2}$$

$$\Rightarrow \quad x' = \frac{\pm cy}{\sqrt{1-c^2y^2}}$$

$$\Rightarrow \quad x = \pm\frac{1}{c}\sqrt{1-c^2y^2} + c_2$$

$$\Rightarrow \quad c^2(x-c_1)^2 = 1-c^2y^2$$

$$\Rightarrow \quad (x-c_2)^2 + y^2 = \frac{1}{c^2}$$

(Equation of the circle with center $(c_2, 0)$ & radius $\frac{1}{c}$)

Page 393 Problem 15

we want to minimize $\int ds$ & $ds = \sqrt{dr^2 + r^2 d\theta^2}$

$$\int ds = \int \sqrt{1 + r^2 \theta'^2}\, dr$$

$$F(r, \theta, \theta') = \sqrt{1 + r^2 \theta'^2}$$

Euler Eq $\quad \frac{d}{dr}\frac{\partial F}{\partial \theta'} - \frac{\partial F}{\partial \theta} = 0$

$\frac{\partial F}{\partial \theta} = 0$ & $\frac{\partial F}{\partial \theta'} = \frac{r^2 \theta'}{\sqrt{1 + r^2 \theta'^2}}$

Euler Eq $\Rightarrow \frac{r^2 \theta'}{\sqrt{1 + r^2 \theta'^2}} = c$ $\quad$ c is a constant

$$\Rightarrow r^4 \theta'^2 = c^2 (1 + r^2 \theta'^2)$$

$$\Rightarrow \theta' = \frac{\pm c}{r (r^2 + c^2)^{1/2}}$$

$$\Rightarrow \theta = \pm \int \frac{c}{r (r^2 + c^2)^{1/2}}\, dr$$

let $r^2 + c^2 = u^2$ so $r\,dr = u\,du$

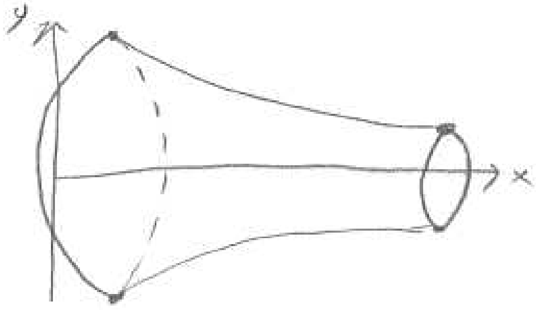
$$\Rightarrow \theta = \pm \int \frac{c \cancel{u}\, du}{r^2 \cdot \cancel{u}} = \pm \int \frac{c\, du}{u^2 - c^2}$$

$$\Rightarrow \theta = \mp \frac{\operatorname{Arctanh}\left(\frac{u}{c}\right)}{c} + c_2$$

$$\Rightarrow \Theta = \mp \frac{\operatorname{Arctanh}\left(\frac{\sqrt{r^2+c^2}}{c}\right)}{c} + c_2$$

Isoperimetric Problems

Page 403 Problem 1



$\int ds = l$ is constant

& Surf Area $= \int 2\pi y \, ds$ where $ds = \sqrt{1+x'^2}\, dy$

so we want to minimize $I = \int y\sqrt{1+x'^2}\, dy$

By Langrange multiplier method we need to make stationary the integral

$$\int \left(y\sqrt{1+x'^2} + \lambda\sqrt{1+x'^2}\right) dy$$

$$= \int (y+\lambda)\sqrt{1+x'^2}\, dy$$

the integrand is the same as the integrand in problem (12 Page 393) except we have $y+\lambda$ instead of $\frac{1}{y}$. so by following the same procedure we get

$$x' = \frac{\frac{c}{y+\lambda}}{\sqrt{1-\frac{c^2}{(y+\lambda)^2}}} \Rightarrow x = c\ln\left(y+\lambda+\sqrt{(y+\lambda)^2-c^2}\right) + c_2$$

where c & c_2 are constant