Solutions of H.W #2

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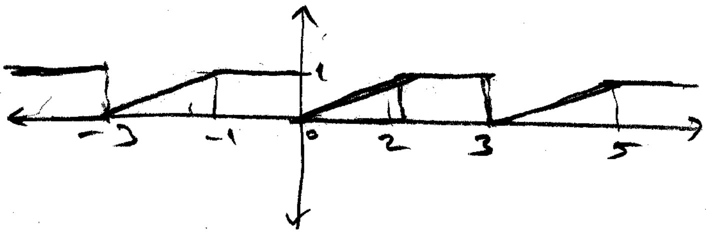
(20) $f(x) = \begin{cases} x/2 & 0<x<2 \\ 1 & 2<x<3 \end{cases}$



expand in complex series.

$$c_n = \frac{1}{2l}\int_0^{2l} f(x)\, e^{-\frac{in\pi}{l}x}\,dx \qquad \text{where } l = 3/2$$

so

$$c_n = \underbrace{\frac{1}{3}\int_0^2 \frac{x}{2} e^{-\frac{i2n\pi x}{3}}\,dx}_{\mathrm{I}} + \underbrace{\frac{1}{3}\int_2^3 e^{-\frac{i2n\pi x}{3}}\,dx}_{\mathrm{II}}$$

I is

integral	derivative	
$e^{-\frac{i2n\pi}{3}x}$	$x/2$	$+$
$\frac{3i}{2n\pi} e^{-\frac{i2n\pi}{3}x}$	$1/2$	$-$
$-\frac{9}{4n^2\pi^2} e^{-\frac{i2n\pi}{3}x}$	0	

$$\mathrm{I} = \left(\frac{3i}{2n\pi}\,\frac{x}{2}\,.\,e^{-\frac{i2n\pi}{3}x} + \frac{9}{8n^2\pi^2} e^{-\frac{i2n\pi}{3}x}\right)\Bigg|_0^2$$

$$\mathrm{I} = \frac{3i}{2n\pi} e^{-\frac{i4n\pi}{3}} + \frac{9}{8n^2\pi^2} e^{-\frac{i4n\pi}{3}} - \frac{9}{8n^2\pi^2}$$

$$\mathrm{II} = \frac{3i}{2n\pi} e^{-\frac{i2n\pi x}{3}}\Bigg|_2^3 = \frac{3i}{2n\pi} - \frac{3i}{2n\pi} e^{-\frac{i4n\pi}{3}}$$

So $c_n = \frac{1}{3}(I + II) =$

$$c_n = \frac{3i}{2n\pi} e^{-\frac{i4n\pi}{3}} + \frac{3}{8n^2\pi^2} e^{-\frac{i4n\pi}{3}} - \frac{3}{8n^2\pi^2} + \frac{i}{2n\pi} - \frac{i}{2n\pi} e^{-\frac{i4n\pi}{3}}$$

for $n = 3k$

$$c_{3k} = \frac{i}{6k\pi} + \cancel{\frac{3}{8\times 9k^2\pi^2}} - \cancel{\frac{3}{8\times 9k^2\pi^2}} + \cancel{\frac{i}{6\pi}} - \cancel{\frac{i}{6\pi}}$$

$$c_{3k} = \frac{1}{6k\pi} i$$

for $n = 3k+1$ $\qquad e^{-\frac{4\pi(3k+1)i}{3}} = e^{-\frac{8\pi k i}{3}} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

$$c_{3k+1} = \frac{i}{2(3k+1)\pi}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) + \frac{3}{8(3k+1)^2\pi^2}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) - \frac{3}{8(3k+1)^2\pi^2}$$

$$+ \frac{i}{2(3k+1)\pi} - \frac{i}{2(3k+1)\pi}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)$$

$$= \cancel{-\frac{i}{4(3k+1)\pi}} \cancel{+ \frac{\sqrt{3}}{4(3k+1)\pi}} - \frac{3}{16(3k+1)^2\pi^2} - \frac{3\sqrt{3}}{16(3k+1)^2\pi^2} i - \frac{3}{8(3k+1)^2\pi^2}$$

$$+ \frac{i}{2(3k+1)\pi} + \cancel{\frac{i}{4(3k+1)\pi}} \cancel{- \frac{\sqrt{3}}{4(3k+1)\pi}}$$

$$= -\frac{9}{16(3k+1)^2\pi^2} + \left(\frac{1}{2(3k+1)\pi} - \frac{3\sqrt{3}}{16(3k+1)^2\pi^2}\right) i$$

$n = 3k+1 \qquad e^{-\frac{4\pi(3k+1)i}{3}} = e^{-\frac{4\pi}{3}i} = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

then $c_n = \dfrac{-9}{16(3k+1)^2\pi^2} + \left(\dfrac{1}{2(3k+1)\pi} + \dfrac{3\sqrt{3}}{16(3k+1)^2\pi^2}\right)i$

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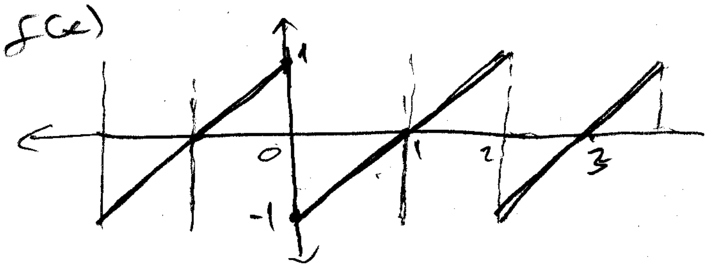
(12) $f(x) = \begin{cases} x+1 & -1<x<0 \\ x-1 & 0 \leq x < 1 \end{cases}$



it is an odd function

so all a_n terms are zero, and to find b_n terms it will be easier to take the integral from 0 to 2.

then $b_n = 1 \displaystyle\int_0^2 \sin(n\pi x).(x-1)\,dx$

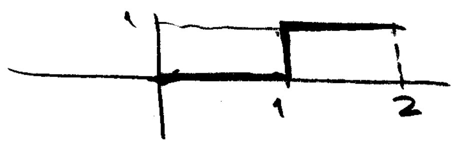
integral	derivative	
$\sin(n\pi x)$	$x-1$	$+$
$-\frac{1}{n\pi}\cos(n\pi x)$	1	$-$
$-\frac{1}{n^2\pi^2}\sin(n\pi x)$	0	

$\Rightarrow b_n = -\frac{1}{n\pi}\cos(n\pi x)(x-1) + \cancel{\frac{1}{n^2\pi^2}\sin(n\pi x)}\Big|_0^2$

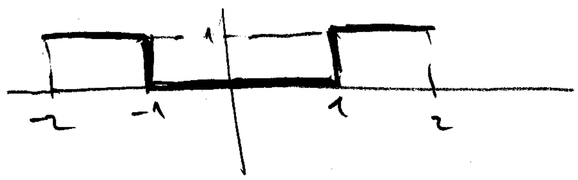
$$b_n = -\frac{1}{n\pi} - \frac{1}{n\pi} = -\frac{2}{n\pi}$$

so $f(x) = \frac{-2}{n\pi} \sin n\pi x - \frac{1}{2\pi} \sin(2n\pi x) - \frac{\pi}{3\pi} \sin(3n\pi x) - \cdots$

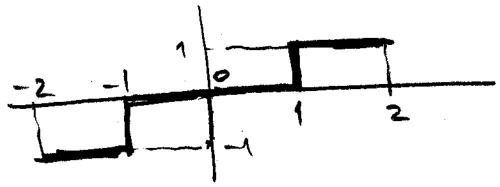
16) $f(x)$.



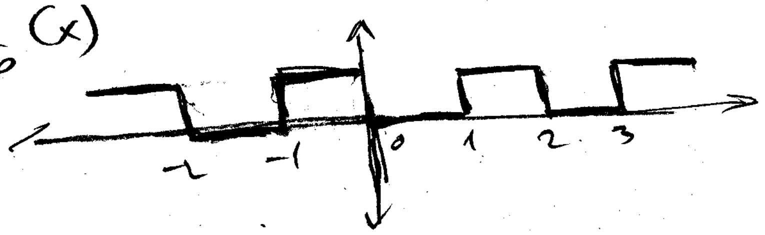
$f_c(x)$



$f_s(x)$



$f_p(x)$



for $f_c(x)$ we have cosine series expantion with period 4

then $a_n = \frac{1}{2} \int_{-2}^{2} f_c(x) \cos\left(\frac{n\pi}{2} x\right) dx = \int_0^2 f_c(x) \cos\left(\frac{n\pi x}{2}\right) dx$

$$= \int_1^2 \cos\left(\frac{n\pi x}{2}\right) dx = \frac{2}{n\pi} \sin\left(\frac{n\pi}{2} x\right)\Big|_1^2 = -\frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

for n is even $a_n = 0$

$a_0 = 1$

$\frac{1}{2}\int_{-2}^{2} f(x)\,dx$

$= \int_1^2 dx = 1$

for n is odd if $n = 4k-1$ $a_n = \frac{2}{n\pi}$

if $n = 4k+1$ $a_n = -\frac{2}{n\pi}$

so

$$f_c(x) = \frac{1}{2} + \sum_{k=0}^{\infty} -\frac{2}{(4k+1)\pi} \cos\left(\frac{(4k+1)\pi x}{2}\right) + \sum_{k=1}^{\infty} \frac{2}{(4k-1)} \cos\left(\frac{(4k-1)\pi x}{2}\right)$$

for $f_s(x)$ we have sine series expantion with period 4.

$$b_n = \frac{1}{2}\int_{-2}^{2} f_s(x) \sin\left(\frac{n\pi}{2}x\right) dx = \int_0^2 f(x) \sin\left(\frac{n\pi}{2}x\right) dx$$

$$= \int_1^2 \sin\left(\frac{n\pi}{2}x\right) dx = -\frac{2}{n\pi} \cos\left(\frac{n\pi}{2}x\right)\Bigg|_1^2$$

$$= -\frac{2}{n\pi}\cos(n\pi) + \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right)$$

If n is odd or $n = 2k+1$.

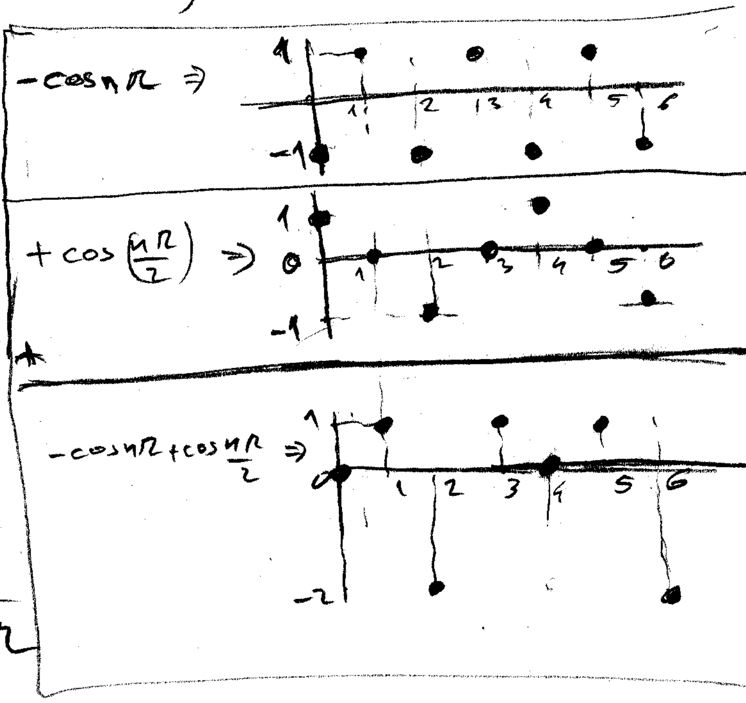
$b_n = \frac{2}{n\pi}$ or $\frac{2}{(2k+1)\pi}$

when n is even if $n = 4k$

$b_n = 0$

if $n = 4k+2$

$b_n = -\frac{2}{n\pi}$ or $\frac{-2}{(4k+2)\pi}$



then $f_s(x)$ is

$$f_s(x) = \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi} \sin\left(\frac{(2k+1)\pi x}{2}\right) + \sum_{k=0}^{\infty} \frac{-2}{(4k+2)\pi} \sin\left(\frac{(4k+2)\pi x}{2}\right)$$

for $f_p(x)$

$$a_n = \int_{-1}^{1} f_p(x) . \cos(n\pi x)\, dx$$

$$\Rightarrow a_n = \int_{-1}^{0} \cos(n\pi x) dx = \frac{1}{n\pi} \sin(n\pi x)\Big|_{-1}^{0} = 0$$

$$a_0 = \int_{-1}^{1} f_p(x)\, dx = \int_{-1}^{0} dx = 1$$

$$b_n = \int_{-1}^{1} f_p(x) . \sin(n\pi x)\, dx = \int_{-1}^{0} \sin(n\pi x) dx$$

$$= -\frac{1}{n\pi} \cos(n\pi x)\Big|_{-1}^{0} = -\frac{1}{n\pi} - \frac{1}{n\pi} = -\frac{2}{n\pi}$$

$$f_p(x) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(n\pi x) .$$

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(5) Parseval's Theorem says that if $f(x)$ is $2l$ periodic function then

$$\frac{1}{2l}\int_{-l}^{l} (f(x))^2 dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$ where a_n & b_n'

are fourier coefficients

Then we know from problem 3.6 that

$$f(x) = \begin{cases} -1 & -l<x<0 \\ 1 & 0<x<l \end{cases}$$ has a fourier expansion

$$f(x) = \frac{4}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \cdots\right)$$

so a_0 & a_n's are zero then according to Parseval's Theorem.

$$\frac{1}{2l}\int_{-l}^{l}(f(x))^2dx = \frac{1}{2}\left(\left(\frac{4}{\pi}\right)^2 + \left(\frac{4}{\pi}\cdot\frac{1}{3}\right)^2 + \left(\frac{4}{\pi}\frac{1}{5}\right)^2 + \cdots\right)$$

$$= \frac{1}{2}\times\frac{16}{\pi^2}\Big(\underbrace{1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots}_{S}\Big)$$

$$\Rightarrow S = \frac{\pi^2}{16l}\int_{-l}^{l}(f(x))^2dx \qquad f(x)^2 = 1 \text{ for } -l<x<l$$

$$= \frac{\pi^2}{16l}\int_{-l}^{l}dx = \frac{\pi^2}{16l}\cdot 2l$$

$$\Rightarrow S = \frac{\pi^2}{8} \quad //$$

(9) In problem 5.11 fourier series of

$$f(x) = \begin{cases} 0 & -\pi<x<0 \\ \sin x & 0<x<\pi \end{cases}$$ is given as

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \cdots\right)$$

Let's calculate average of $f(x)^2$ over one period.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x))^2dx = \frac{1}{2\pi}\int_0^{\pi}\sin^2 x\,dx = \frac{1}{2\pi}\int_0^{\pi}\left(\frac{1}{2}-\frac{\cos 2x}{2}\right)dx$$

$$= \frac{1}{2\cancel{\pi}}\times\frac{\cancel{\pi}}{2} = \frac{1}{4}$$

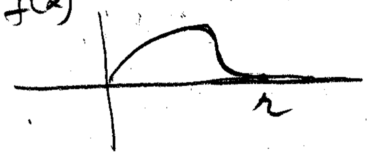
Then according to parsevals theorem

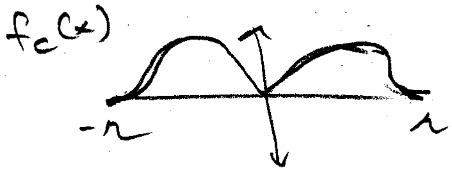
$$\text{average} = \frac{1}{4} = \left(\frac{1}{\pi}\right)^2 + \frac{1}{2}\times\left(\frac{1}{2}\right)^2 + \frac{1}{2}\left(\left(\frac{2}{\pi}\cdot\frac{1}{3}\right)^2+\left(\frac{2}{\pi}\cdot\frac{1}{15}\right)^2+\left(\frac{2}{\pi}\cdot\frac{1}{35}\right)^2+\cdots\right.$$

$$\Rightarrow \frac{1}{4} = \frac{1}{\pi^2} + \frac{1}{8} + \frac{1}{2}\times\frac{4}{\pi^2}\underbrace{\left(\frac{1}{3^2}+\frac{1}{15^2}+\frac{1}{35^2}+\cdots\right)}_{S}$$

$$\Rightarrow S = \left(\frac{1}{8}-\frac{1}{\pi^2}\right)\times\frac{2\pi^2}{4} = \frac{\pi^2}{16}-\frac{1}{4}$$

Pr. 4





parseval's theorem

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(f_c(x))^2dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum a_n^2$$

but $\frac{1}{2\pi}\int_{-\pi}^{\pi}(f_c(x))^2dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2dx$

then

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}a_n^2$$

$$\Rightarrow \frac{2}{\pi}\int_0^{\pi}(f(x))^2 = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}a_n^2$$

Prob 5

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx \quad \text{, then}$$

$$\hat{f}(\omega-k) = \int_{-\infty}^{\infty} f(x)\, e^{-i(\omega-k)x}\, dx$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{ikx} \cdot e^{-i\omega x}\, dx$$

but the last expression is identical with the fourier transform of $f(x) \cdot e^{ikx}$.

then $\quad F[e^{ikx} f(x)] = \hat{f}(\omega-k)$

Prob 6

Let's take the an coefficients of fourier series.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L}x\right) dx$$

we can use integration by parts method for this integral by calling $u = f(x)$ & $dv = \cos\left(\frac{n\pi}{L}x\right)dx$

$du = f'(x)\,dx \qquad v = \frac{L}{n\pi}\sin\left(\frac{n\pi}{L}\right)$, then

$$a_n = \frac{1}{L}\left(\left(f(x)\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{-L}^{L} - \int_{-L}^{L}\frac{L}{n\pi} f'(x)\sin\left(\frac{n\pi}{L}\right)dx\right)$$

The first term in the right hand side is zero since $\sin(n\pi)$ is zero, and integral term also tends to zero as n goes to infinity. (for bn's and cn's the method is the same)