Math 208, Spring 2013, Quiz # 3

You have 40 minutes.

Name, Last Name | Student ID Number | Signature

Problem 1 (10 points) Give the definition or precise statement of the following.

1.a) A function $f : \mathbb{R} \to \mathbb{R}$ that is differentiable at a point $x_0 \in \mathbb{R}$:

1.b) Rolle's Theorem:

1.c) Cauchy Mean-Value Theorem:

1.d) A sequence $\{f_n\}$ of functions $f_n : \mathbb{R} \to \mathbb{R}$ that converges pointwise:

Problem 2 (6 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined on $\mathbb{R}$ and $c \in \mathbb{R}^+$ be such that $|f(u) - f(v)| \leq c(u - v)^2$ for all $u, v \in \mathbb{R}$. Show that f is a constant function.

We will show that $\forall x \in \mathbb{R}$, $f'(x) = 0$
which implies that f is constant.
So let $x \in \mathbb{R}$ and x_n be a sequence in $\mathbb{R} \setminus \{x\}$
such that $\lim\limits_{n\to\infty} x_n = x$.
Then $\left|\dfrac{f(x_n) - f(x)}{x_n - x}\right| \leq |x_n - x|$.

So by comparison lemma $\lim\limits_{n\to\infty} \left|\dfrac{f(x_n) - f(x)}{x_n - x}\right| = 0$
which implies that $f'(x) = 0$.

Problem 3 (6 points) Let $f : (-1,1) \to \mathbb{R}$ be a three-times differentiable function such that $f(0) = f'(0) = f''(0) = 0$ and $f^{(3)}(x) \in (-1,3)$ for all $x \in (-1,1)$. Show that $|f(x)| < \frac{1}{2}$ for all $x \in (-1,1)$.

By theorem 4.24 for each point $x \neq 0$ in $(-1,1)$ there exist z strictly between x and x_0 such that

$$f(x) = \frac{f^{(3)}(z)}{3!}(x-0)^3$$

Then $|f(x)| = \frac{|f^{(3)}(z)|}{6}|x|^3 \leq \frac{3}{6}\cdot 1 = \frac{1}{2}$ since $|x|^3 \leqslant 1$ for $x \in (-1,1)$

which completes the proof

Problem 4 (8 points) Let $D \subseteq \mathbb{R}$, $f, g : D \to \mathbb{R}$ and (for all $n \in \mathbb{N}$) $f_n, g_n : D \to \mathbb{R}$ be functions such that $\{f_n\}$ and $\{g_n\}$ converge uniformly to f and g respectively. Show that $\{f_n + g_n\}$ converges to $f + g$ uniformly.

Let $\varepsilon > 0$. Then $\exists\ N_1, N_2 \in \mathbb{N}$ such that

$\forall x \in D \quad |f_n(x) - f(x)| < \varepsilon/2$

and

$\forall x \in D \quad |g_n(x) - g(x)| < \varepsilon/2$

Let $N := \max\{N_1, N_2\}$. Then for $n \geq N$ and $x \in D$

$$|(f_n(x) + g(x)) - (f(x) + g(x))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

$\blacksquare$