Math 208, Spring 2013, Quiz # 2

You have 40 minutes.

Name, Last Name	Student ID Number	Signature

Problem 1 (10 points) Give the definition of the following terms.

1.a) Bounded Sequence:

1.b) Subsequence of a sequence:

1.c) Convergent sequence:

1.d) Sequentially compact subset:

1.e) Continuous function:

Problem 2 (6 points) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions defined on $\mathbb{R}$ such that for all $x \in \mathbb{R}$, $g(x) = |f(x)|$. Show that if f is continuous, then so is g.

Let $x \in Dom(g) = \mathbb{R}$ and x_n be a sequence of real numbers converging to $x \in \mathbb{R}$. Then

$$|g(x_n) - g(x)| = \big|\, |f(x_n)| - |f(x)| \,\big| \leq |f(x_n) - f(x)|.$$

Since f is continuous, it is continuous at every $x \in Dom(f) = \mathbb{R}$.

So $\lim_{n\to\infty} |f(x_n) - f(x)| = 0$. Then, by comparison lemma

$$\lim_{n\to\infty} |g(x_n) - g(x)| = 0$$

which is equivalent to saying that $\lim_{n\to\infty} g(x_n) = g(x)$.

This implies that g is continuous at x. Since $x \in Dom(g) = \mathbb{R}$ is arbitrary, we conclude that g is continous.

Problem 3 (7 points) Prove that the interval $(-\infty, 0]$ is a closed subset of $\mathbb{R}$.

A set $A \subseteq \mathbb{R}$ is said to be closed, if for any sequence x_n in A converging to a point $x \in \mathbb{R}$, we have $x \in A$.

So let x_n be a sequence in $(-\infty, 0]$ and assume that for some $x \in \mathbb{R}$, $\lim_{n\to\infty} x_n = x$. If we can show that $x \in (-\infty, 0]$, this will imply $(-\infty, 0]$ is closed.

Assume by contradiction $x > 0$. Then for $\varepsilon = \frac{x}{2} > 0$,

$$|x_n - x| < \varepsilon \Leftrightarrow |x_n - x| < \frac{x}{2} \Longleftrightarrow x - x_n < \frac{x}{2}$$

here we used $x > 0$ and $x_n \in (-\infty, 0]$

So $x_n > \frac{x}{2} > 0$. Since $\lim_{n\to\infty} x_n = x$ and $\varepsilon = \frac{x}{2} > 0$, this last inequality has to be satisfied for n large enough but this is not possible since $x_n \in (-\infty, 0]$. Hence $x \leq 0$ and $x \in (-\infty, 0]$.

Problem 4 (7 points) Let $\{a_n\}$ be a monotonically decreasing sequence that has a convergent subsequence. Prove that $\{a_n\}$ converges.

Let a_n be a monotonically decreasing sequence and a_{n_k} be a convergent subsequence. We will show that a_n is bounded from below.

As a_{n_k} is convergent, it is bounded, in particular $\exists M \in \mathbb{R}$ such that $a_{n_k} > M$ for all $k \in \mathbb{N}$.

Now let $m \in \mathbb{N}$. Then $\boxed{m \leq n_m.}$ $\Rightarrow$ (You have to understand this very well.)

Since a_n is a decreasing sequence

$$a_m \geq a_{n_m} > M$$

Since $m \in \mathbb{N}$ is arbitrary, this implies that a_n is bounded from below. Hence, by monotone convergence lemma (decreasing case) a_n is convergent.