Math 208: Midterm Exam 2

Spring 2013

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	SOLUTIONS

- You have 80 minutes.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the exam within the first 10 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)
- (Optional) Grade your own work out of 100. Record your estimated grade here:

Estimated Grade:	

If your expected grade and actual grade will turn out to differ by 9 points or less, you will be given the highest of the two.

To be filled by the grader:

Actual Grade:	
Adjusted Grade:	

Problem 1. (20 points) Let $\{a_n\}$ be a Cauchy sequences in $\mathbb{R}$. Show that it is a bounded sequence.

Let a_n be a Cauchy sequence in $\mathbb{R}$ and
Let $\varepsilon > 0$. Then $\exists$ $N \in \mathbb{N}$ s.t. for $m,n \geq N$

$$|a_n - a_m| < \varepsilon$$

In particular for all $m \geq N$

$$|a_N - a_m| < \varepsilon.$$

This implies that

$$|a_m| - |a_N| < \varepsilon$$

So for all $m \geq N$

$$|a_m| < \varepsilon + |a_N|$$

Let $M := \max\{|a_1|, \ldots, |a_{N-1}|, \varepsilon + |a_N|\}$
Then for all $m \in \mathbb{N}$

$$|a_m| < M$$

so that a_n is a bounded sequence.

$\boxtimes$

Problem 2. Let $n \in \mathbb{N}$ and $A \subseteq \mathbb{R}^n$.

2.a (5 points) Give the definition of the boundary points and the boundary of A.

Let $A \subseteq \mathbb{R}^n$ and $x \in A$. If for all $\varepsilon > 0$, we have $B_\varepsilon(x) \cap A \neq \emptyset$ and $B_\varepsilon(x) \cap A^c \neq \emptyset$, we say that x is a boundary point of A. Boundary of A is the set consisting of boundary points of A.

2.b) (20 points) Show that the boundary of A is a closed subset of $\mathbb{R}^n$.

1st proof: Let x_n be a convergent sequence in boundary of A and $a := \lim_{n\to\infty} x_n$. Let $\varepsilon > 0$. We will show that a is also ~~a~~ boundary point.

i) $B_\varepsilon(a) \cap A \neq \emptyset$:

Suppose that $B_\varepsilon(a) \cap A = \emptyset$. Since $\lim_{n\to\infty} x_n = a$, $\exists N \in \mathbb{N}$ s.t. $\|x_N - a\| < \varepsilon$, i.e. $x_N \in B_\varepsilon(a)$.

Since $B_\varepsilon(a)$ is open and $x_N \in B_\varepsilon(a)$, there exist $r > 0$ s.t. $B_r(x_N) \subset B_\varepsilon(a)$. This, together with $B_\varepsilon(a) \cap A = \emptyset$, implies that $B_r(x_N) \cap A = \emptyset$. Since x_N is a boundary point of A, this is not possible. Hence $B_\varepsilon(a) \cap A \neq \emptyset$.

ii) $B_\varepsilon(a) \cap A^c \neq \emptyset$:

Change A with A^c in (i).

Since ε is arbitrary, a is a boundary point of A. ▨

2nd proof: $\mathbb{R}^n = \text{int}(A) \cup \text{bd}(A) \cup \text{ext}(A)$.

Since $\text{int}(A)$ and $\text{ext}(A)$ are open sets (you need to show this if you did not prove it in CLASS), we know that $\text{int}(A) \cup \text{ext}(A)$ is also open. Since $\text{bd}(A) = (\text{int}(A) \cup \text{ext}(A))^c$, as a complement of an open set, $\text{bd}(A)$ is closed. ▨

2nd proof: Since dom$(f) = \mathcal{O}$ which is open, we know that inverse image of an ~~[illegible]~~ open set under the continuous function f is also open. Since $(0,1)$ is an open set and $A = f^{-1}((0,1))$, A is open.

Problem 3 Let $n \in \mathbb{N}$.

Don't forget that you can use only theorems which you proved in class!!!

3.a (5 points) Give the definition of an open subset of $\mathbb{R}^n$.

Let $\mathcal{O} \subseteq \mathbb{R}^n$. If for all $x \in \mathcal{O}$, there exist an $\varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq \mathcal{O}$, $\mathcal{O}$ is said to be an open set.

3.b (10 points) Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$, and $f : \mathcal{O} \to \mathbb{R}$ be a continuous function defined on $\mathcal{O}$. Show that $\{\mathbf{u} \in \mathcal{O} \mid 0 < f(\mathbf{u}) < 1\}$ is an open subset of $\mathbb{R}^n$.

1st proof: Let $A := \{u \in \mathcal{O} \mid 0 < f(u) < 1\}$ and $x \in A$. Then $0 < f(x) < 1$.

Let $\varepsilon := \min\{f(x), 1 - f(x)\}$.

~~Then $B_\varepsilon(f(x)) \subseteq$ [illegible]~~

Since f is continuous at x, by ε-δ characterization of continuity, there exist $\delta > 0$ s.t. for all $y \in \mathcal{O}$ satisfying $\|y - x\| < \delta$, we have $\|f(y) - f(x)\| < \varepsilon$. Using the open ball notation, this is equivalent to the following:

$$y \in B_\delta(x) \cap \mathcal{O} \implies f(y) \in B_\varepsilon(f(x))$$

Don't forget that this norm is indeed the absolute value, i.e. $\|f(y) - f(x)\| = |f(y) - f(x)|$

As an intersection of open sets $B_\delta(x) \cap \mathcal{O}$ is open.

Also for $y \in B_\delta(x) \cap \mathcal{O}$,

$$|f(y) - f(x)| < \varepsilon = \min\{f(x), 1 - f(x)\}$$

so that $f(y) \in (0,1)$. So $B_\delta(x) \cap \mathcal{O} \subset A$.

Hence x lies in the open set $B_\delta(x) \cap A$ which lies in A. From this it follows easily that $x \in \text{int}(A)$. Since x is arbitrary, A is an open set. $\blacksquare$

Problem 4

4.a (5 points) Give the definition of a pathwise connected subset of $\mathbb{R}^n$.

Let $A \subset \mathbb{R}^n$. Then A is said to be a pathwise connected set if for any two points $x, y \in A$ there exist a <u>continuous</u> function $\gamma : [a,b] \to \mathbb{R}^n$ which satisfies

(i) $\gamma(a) = x$, $\gamma(b) = y$

(ii) $\gamma([a,b]) \subset A$.

4.b (15 points) Let $n \in \mathbb{N}$, and A and B be pathwise connected subsets of $\mathbb{R}^n$ having a nonempty intersection. Show that the union of A and B is also pathwise connected.

Let $x, y \in A \cup B$ and $z \in A \cap B$ (since $A \cap B \neq \emptyset$).

Since A and B are pathwise-connected, there exist parametrized paths $\gamma_1 : [a,b] \to A$ and $\gamma_2 : [c,d] \to B$ such that

$$\gamma_1(a) = x,\ \gamma_1(b) = z,\ \gamma_2(c) = z,\ \gamma_2(d) = y.$$

Let $\varphi : [0,1] \to \mathbb{R}$ be defined as

$$\varphi(t) = bt + a(1-t)$$

and $\psi : [1,2] \to \mathbb{R}$ be defined as

$$\psi(t) = d(t-1) + c(2-t)$$

Note that ψ and φ are continuous functions (polynomial functions are continuous)

Then $\tilde{\gamma}_1 := \gamma_1 \circ \varphi$ and $\tilde{\gamma}_2 := \gamma_2 \circ \psi$ are also continuous as compositions of continuous functions. Also

$\tilde{\gamma}_1(0) = x$, $\tilde{\gamma}_1(1) = z$, $\tilde{\gamma}_2(1) = z$, $\tilde{\gamma}_2(2) = y$.

~~[illegible]~~ and $\tilde{\gamma}_1([0,1]) \subset A$ and $\tilde{\gamma}_2([0,1]) \subset B$.

Define $\gamma : [0,2] \to A \cup B$ as

$$\gamma(t) := \begin{cases} \tilde{\gamma}_1(t) & \text{for } 0 \le t \le 1 \\ \tilde{\gamma}_2(t) & \text{for } 1 \le t \le 2 \end{cases}$$

Since $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1) = z$, γ is continuous and its image lies in $A \cup B$ so that γ is a parametrized path from x to y. Since x, y are arbitrary $A \cup B$ is path-con. $\square$

Problem 5 (20 points) Let $n \in \mathbb{N}$, $A \subseteq \mathbb{R}^n$, and $f : A \to \mathbb{R}$ be a continuous function defined on A such that $f(A) = \{-1, 1\}$. Show that A is a disconnected subset of $\mathbb{R}^n$.

<u>1st proof</u>: If you proved in CLASS that image of a connected set under a continuous function is connected, then you could use the following argument:

If A were connected, $f(A)$ would be a connected set. $\boxed{\text{Connected sets in } \mathbb{R} \text{ are intervals.}}$ $\{-1, 1\}$ is not an interval since $-1 < 0 < 1$ and $0 \notin \{-1, 1\}$. So A is not connected. QED

You could use this if you proved this in class.

<u>2nd proof</u>: Let $B := f^{-1}(\{-1\})$ and $C := f^{-1}(\{1\})$.
Since f is continuous, for all $x \in B$, there exist $\delta_x > 0$ s.t. $f(B_{\delta_x}(x) \cap A) \subset B_{1/2}(-1)$.

Similarly for all $y \in C$, there exist $\delta_y > 0$ s.t. $f(B_{\delta_y}(y) \cap A) \subset B_{1/2}(1)$.

Let $U := \bigcup_{x \in B} B_{\delta_x}(x)$, and $V := \bigcup_{y \in C} B_{\delta_y}(y)$.
As unions of open sets U and V are open.
Also $B \subset U$ and $C \subset V$ so that $A = B \cup C \subset U \cup V$.
So $B = A \cap U \neq \emptyset$ and $C = A \cap V \neq \emptyset$. Lastly, for any two points $x \in B = A \cap U$ and $y \in C = A \cap V$

$$(B_{\delta_x}(x) \cap A) \cap (B_{\delta_y}(y) \cap A) = \emptyset.$$

Hence $\underbrace{(U \cap A) \cap (V \cap A) = \emptyset}_{\text{This condition is given in the definition of the book.}}$ and this completes the proof.

<u>Exc</u>: This is equivalent to the definition given in the class.