Math 208 - Spring 2010
Midterm # 1, Solutions.

Problem 1. State Completeness Axiom for $\mathbb{R}$ and use it to prove: $\forall r \in \mathbb{R}^+, \exists n \in \mathbb{N}, r < n$. (20 points)

Completeness Axiom: let S be a subset of $\mathbb{R}$ that is bounded above. Then S has a least upper bound (supremum) in $\mathbb{R}$.

Proof of $\forall r \in \mathbb{R}^+, \exists n \in \mathbb{N}, r < n$: Suppose by contradiction $\exists r \in \mathbb{R}, \forall n \in \mathbb{N}, r \geq n \Rightarrow \mathbb{N}$ is bounded above in $\mathbb{R}$.
By Completeness axiom, $\mathbb{N}$ has a supremum in $\mathbb{R}$, call it $m \Rightarrow m - \frac{1}{2}$ is not an upper bound of $\mathbb{N}$ in $\mathbb{R} \Rightarrow \exists N \in \mathbb{N} \ni N > m - \frac{1}{2}$ (1)
$N \in \mathbb{N} \Rightarrow N+1 \in \mathbb{N}$ because $\mathbb{N}$ is inductive.
(1) $\Rightarrow N+1 > m + \frac{1}{2} > m$ by positivity axiom
$N+1 \in \mathbb{N} \wedge N+1 > m \Rightarrow m$ is not an upper bound of $\mathbb{N}$ in $\mathbb{R}$. This is a contradiction. ▨

Problem 2. Give the definition of a sequentially compact subset of $\mathbb{R}$ and state The Sequential Compactness (Bolzano-Weierstrass) theorem. (10 points)

$A \subseteq \mathbb{R}$ is said to be sequentially compact if every sequence in A has a convergent subsequence.

Thm: $\forall a, b \in \mathbb{R}$ with $a < b$, $[a, b]$ is sequentially compact.

Problem 3. Prove that a monotone increasing sequence in $\mathbb{R}$ that has a convergent subsequence converges. (20 points)

Let $\{a_n\}$ be a monotonically increasing sequence that has a convergent subsequence $\{a_{n_k}\}$ $\Rightarrow$

$\exists a_0 \in \mathbb{R}, \ \forall \epsilon > 0, \ \exists N \in \mathbb{N}, \ \forall k \geq N, \ |a_{n_k} - a_0| < \epsilon$

$\Leftrightarrow -\epsilon < a_{n_k} - a_0 < \epsilon \Rightarrow a_0 - \epsilon < a_{n_k} < a_0 + \epsilon$

$\{a_n\}$ is monotonically increasing and $\{n_k\}$ is strictly increasing $\Rightarrow$

$\forall m \geq N, \ n_m \geq m \Rightarrow n_m \geq N \hookrightarrow a_{n_m} < a_0 + \epsilon$

$\forall m < N \leq n_N \Rightarrow a_m \leq a_{n_N} < a_0 + \epsilon$

$\Rightarrow \forall m \in \mathbb{N}, \ a_m < a_0 + \epsilon \Rightarrow \{a_m\}$ is bounded above. But it is also monotonically increasing these imply that it must be convergent.

$\boxtimes$

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, $a, b \in \mathbb{R}$ and $a < b$ Prove that the image of $[a, b]$ under f is bounded above. (20 points)

Suppose $f[a,b]$ is not bounded above.

$\forall M \in \mathbb{R}^+, \exists x \in [a,b], |f(x)| > M$

$\Rightarrow$ $\boxed{\forall n \in \mathbb{N}, \exists x_n \in [a,b], |f(x_n)| > n}$ (1)

$\{x_n\}$ is a sequence in $[a,b]$ $\Rightarrow$ By (B-W thm) it has a convergent subsequence $\{x_{n_k}\}$.

f is continuous $\Rightarrow$ $\{f(x_{n_k})\}$ converges to $f(x_0)$ where $x_0 = \lim_{k\to\infty} x_{n_k}$. $\Rightarrow$ $\{f(x_{n_k})\}$ is a bounded sequence

$\Rightarrow \exists K \in \mathbb{R}^+, \forall k \in \mathbb{N}, |f(x_{n_k})| \leq K$

$\Downarrow$

$\boxed{|f(x_{n_K})| \leq K}$ (2)

(1) $\Rightarrow$ $n_K < |f(x_{n_K})|$

But $\{n_k\}$ is strictly increasing $\Rightarrow$ $K \leq n_K$.

$\Rightarrow$ $\boxed{K < |f(x_{n_K})|}$ (3)

(3) Contradicts (2) $\Rightarrow$ $\blacksquare$

Problem 5. Let I be an open interval and $f : I \to \mathbb{R}$ be a differentiable function with domain I. Suppose that $\forall x \in I$, $f'(x) > 0$. Prove that f is strictly increasing. (15 points)

$\forall u, v \in I$, with $u < v$ the restriction of f on $[u, v]$ is continuous and its restrict to (u, v) is differentiable $\Rightarrow$ By mean value theorem $\exists c \in (u, v)$ such that $f'(c) = \dfrac{f(v) - f(u)}{v - u}$

$c \in I \Rightarrow f'(c) > 0 \Rightarrow \dfrac{f(v) - f(u)}{v - u} > 0$ $v - u > 0 \Rightarrow$

$f(v) - f(u) > 0 \Rightarrow f(v) > f(u)$ then f is strictly increasing. ☒

Problem 6. Let $\forall n \in \mathbb{N}$, $f_n : [0, 1] \to \mathbb{R}$ be the function defined by $\forall x \in [0, 1]$, $f_n(x) := \dfrac{x}{nx + 1}$. Prove that $\{f_n\}$ converges to the (constant function) zero uniformly. (15 points)

For $x = 0$, $f_n(0) = 0$, $\forall n \in \mathbb{N}$

for $x \neq 0$, $f_n(x) = \dfrac{1}{n + \frac{1}{x}}$, $\forall n \in \mathbb{N}$

Now $\forall \epsilon > 0$, $\exists N \in \mathbb{N} \ni \frac{1}{\epsilon} < N \Rightarrow \epsilon > \frac{1}{N}$

$\forall n \geq N$, $\frac{1}{n} \leq \frac{1}{N} < \epsilon$ $\Rightarrow$

For $x > 0$, $n + \frac{1}{x} > n \Rightarrow \dfrac{1}{n + \frac{1}{x}} < \frac{1}{n} < \epsilon \Rightarrow \left| \dfrac{1}{n + \frac{1}{x}} \right| < \epsilon$

Also $0 < \dfrac{1}{n + \frac{1}{x}}$

$\left| \dfrac{x}{nx + 1} \right| < \epsilon$

For $x = 0$, $\left| \dfrac{x}{nx + 1} \right| = 0 < \epsilon \Rightarrow \forall x \in [0, 1]$, $\forall \epsilon > 0$ $\exists N \in \mathbb{N}$, $\forall n \geq N$

$|f_n(x)| = \left| \dfrac{x}{nx + 1} \right| < \epsilon \Rightarrow \{f_n\}$ converges uniformly to $f(x)$ where $\forall x \in [0, 1]$, $f(x) = 0$.

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