Solution

Name: Student ID: Signature:

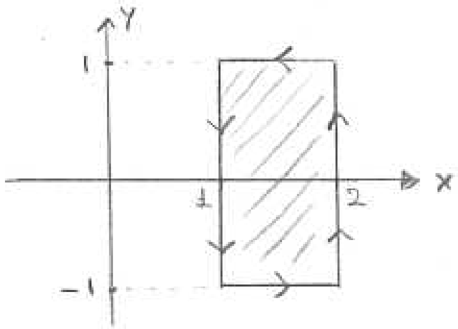
# Math 207: Quiz # 3A

**Fall 2004**

- You have 35 minutes.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask 2 points will be deduced from your grade.

1. Use Green's theorem to compute the line integral $\oint \vec{F} \cdot d\vec{x}$ over the closed curve bounding the dashed region shown in the following figure, where $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\vec{F} = (F_1, F_2)$ and

$$F_1(x,y) = y^2 + \frac{e^{xy}}{x} + e^{x^2}, \qquad F_2(x,y) = x^2 + \frac{e^{xy}}{y} + e^{-y^2}. \qquad \text{(6 points)}$$




$$I := \oint \vec{F} \cdot d\vec{x} = \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy$$

$$\frac{\partial F_2}{\partial x} = 2x + e^{xy} \qquad \frac{\partial F_1}{\partial y} = 2y + e^{xy}$$

$$\Rightarrow \quad I = \int_1^2 dx \int_{-1}^{1} dy\, (2x - 2y)$$

$$= \int_1^2 dx \left[ 2x(y)\Big|_{-1}^{1} - 2\frac{y^2}{2}\Big|_{-1}^{1} \right] \quad \nearrow 0$$

$$= 4 \int_1^2 x\, dx = 2x^2\Big|_1^2 = 2(4-1) = 6$$

2. Let $f : \mathbb{C} \to \mathbb{C}$ and $u, v : \mathbb{R}^2 \to \mathbb{R}$ be functions satisfying for all $z \in \mathbb{C}$, $f(z) = u(x,y) + iv(x,y)$ where $z = x + iy$ and $x, y \in \mathbb{R}$. Show that if $f$ is an entire function and $u$ and $v$ are twice differentiable in $\mathbb{R}^2$, then $\nabla^2 v(x,y) = 0$ for all $x, y \in \mathbb{R}$. (4 points)

$f$ is entire $\Rightarrow$ $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$$\nabla^2 v = \frac{\partial^2}{\partial x^2} v + \frac{\partial^2}{\partial y^2} v = \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} = 0$$

3. Let $f : \mathbb{C} \to \mathbb{C}$ be the function defined by

$$f(z) = \frac{e^{(z^2)} - e^{-(z^2)}}{z^5}.$$

3.a) Find the principal part of the Laurent series for $f$ about $z = 0$. (6 points)

Hint: Recall that for all $w \in \mathbb{C}$, $e^w = 1 + w + w^2/2! + w^3/3! + \cdots$.

$$e^{z^2} = 1 + z^2 + \frac{z^4}{2!} + \frac{z^6}{3!} + \frac{z^8}{4!} + \frac{z^{10}}{5!} + \cdots$$

$$e^{-z^2} = 1 - z^2 + \frac{z^4}{2!} - \frac{z^6}{3!} + \frac{z^8}{4!} - \frac{z^{10}}{5!} \pm \cdots$$

$$\Rightarrow \quad f(z) = \frac{1}{z^5}\left(2z^2 + \frac{2}{3!}z^6 + \frac{2}{5!}z^{10} + \cdots\right)$$

$$= \frac{2}{z^3} + \frac{1}{3}z + \frac{1}{60}z^5 + \cdots$$

So the principal part is $\frac{2}{z^3}$.

3.b) Show that $z = 0$ is a multiple pole and determine its order. (2 points)

The principal part includes one term namely $\frac{2}{z^3}$

So $z = 0$ is a pole of order $\underline{3}$.

3.c) Determine the residue of $f$ at $z = 0$. (2 points)

$\frac{b_1}{z}$ term is absent $\Rightarrow$ $b_1 = R(z_0) = 0$.