Name: Student ID:

Math 207: Quiz # 2B

Fall 2004

- You have 25 minutes.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask 2 points will be deduced from your grade.

---

1. Let $\mathbb{A}$, $\mathbb{B}$, $\mathbb{H}$ and $\mathbb{U}$ be $n \times n$ matrices such that $\mathbb{H}$ is Hermitian and $\mathbb{U}$ is unitary.

1.a) Show that $(\mathbb{B}\mathbb{A})^T = \mathbb{A}^T\mathbb{B}^T$, where T stands for the "transpose".
Hint: Let $\mathbb{M} = \mathbb{B}\mathbb{A}$ and compute the entries of $\mathbb{M}^T$ in terms of the entries of $\mathbb{A}$ and $\mathbb{B}$. (3 points)

$$(\mathbb{M}^T)_{ij} = M_{ji} = \sum_{k=1}^{3} B_{jk} A_{ki}$$

$$(\mathbb{A}^T\mathbb{B}^T)_{ij} = \sum_{k=1}^{3} \mathbb{A}^T_{ik}\mathbb{B}^T_{kj} = \sum_{k=1}^{3} A_{ki}B_{jk} = \sum_{k=1}^{3} B_{jk}A_{ki} = (\mathbb{M}^T)_{ij}$$

$$\Rightarrow \mathbb{A}^T\mathbb{B}^T = \mathbb{M}^T = (\mathbb{B}\mathbb{A})^T \quad \boxed{}$$

1.b) Show that $(\mathbb{B}\mathbb{A})^* = \mathbb{A}^*\mathbb{B}^*$, where $*$ stands for the "Hermitian conjugate". (2 points)

$$(\mathbb{B}\mathbb{A})^* = (\overline{\mathbb{B}\mathbb{A}})^T = (\overline{\mathbb{B}}\,\overline{\mathbb{A}})^T = \overline{\mathbb{A}}^T\,\overline{\mathbb{B}}^T = \mathbb{A}^*\mathbb{B}^* \quad \boxed{}$$

1.c) Show that $\mathbb{U}\mathbb{H}\mathbb{U}^{-1}$ is Hermitian. (5 points)

$$(\mathbb{U}\mathbb{H}\mathbb{U}^{-1})^* = \mathbb{U}^{-1*}(\mathbb{U}\mathbb{H})^* = \mathbb{U}^{-1*}\,\mathbb{H}^*\,\mathbb{U}^*$$

$$= (\mathbb{U}^{-1})^{-1}\,\mathbb{H}\,\mathbb{U}^{-1}$$ (IH is Hermitian, U is unitary $\Rightarrow$ $\mathbb{U}^{-1}$ is unitary)

$$= \mathbb{U}\,\mathbb{H}\,\mathbb{U}^{-1} \quad \boxed{}$$

2. Let a, b, r, s be real numbers and $\mathbb{U} = \begin{pmatrix} 2re^{ia} & s\,e^{-ia} \\ 2re^{ib} & -s\,e^{-ib} \end{pmatrix}$. How should a and b be related so that $\mathbb{U}$ is a unitary matrix? Use the condition that $\mathbb{U}$ is unitary to determine r and s. (10 points)

If U is unitary, $\vec{v}_1 = \begin{pmatrix} 2re^{ia} \\ 2re^{ib} \end{pmatrix}$ & $\vec{v}_2 = \begin{pmatrix} se^{-ia} \\ -se^{-ib} \end{pmatrix}$

must be orthonormal

$$1 = \langle \vec{v}_1, \vec{v}_1 \rangle = 4r^2 + 4r^2 = 8r^2 \Rightarrow$$

$$1 = \langle \vec{v}_2, \vec{v}_2 \rangle = s^2 + s^2 = 2s^2 \Rightarrow$$

$$\boxed{\begin{matrix} r = \pm \frac{1}{\sqrt{8}} \\ s = \pm \frac{1}{\sqrt{2}} \end{matrix}}$$

$$0 = \langle \vec{v}_1, \vec{v}_2 \rangle = \vec{v}_1^{\,*} \vec{v}_2 = (2re^{-ia} \quad 2re^{-ib}) \begin{pmatrix} se^{-ia} \\ -se^{-ib} \end{pmatrix}$$

$$= \underbrace{2rs}_{\neq 0} (e^{-2ia} - e^{-2ib})$$

$$\Rightarrow e^{-2ia} = e^{-2ib} \Rightarrow e^{2i(b-a)} = 1$$

$$\Rightarrow 2(b-a) = 2\pi k \text{ for some } k \in \mathbb{Z}$$

$$\Downarrow$$

$$\boxed{b = a + \pi k}$$

✓

$$\Downarrow$$

$$e^{ib} = e^{ia} e^{\pi k} = \pm e^{ia}$$