Solution

Math 207: Midterm Exam # 1A

Fall 2004

- Write your name and Student ID number in the space provided below and sign.

Student's Name:	
ID Number:	
Signature:	

- You have 80 minutes.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the exam within the first 10 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)
- (Optional) Grade your own work out of 100. Record your estimated grade here:

Estimated Grade:	

If your expected grade and actual grade will turn out to differ by 9 points or less, you will be given the highest of the two.

To be filled by the grader:

Actual Grade:	
Adjusted Grade:	

1. Determine the center and the radius of convergence of the following complex power series.

$$S := \sum_{n=0}^{\infty} \frac{(z-1)^n}{n+1}. \qquad \text{(25 points)}$$

Center is $z = 1$

Consider $\sum_{n=0}^{\infty} \left| \frac{(z-1)^n}{n+1} \right| = \sum_{n=0}^{\infty} \underbrace{\frac{|z-1|^n}{n+1}}_{a_n}$

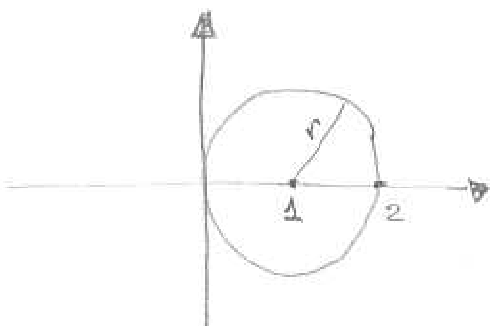
For $z \neq 1$, $a_n > 0$ $\Rightarrow$ apply ratio test

$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \left(\frac{|z-1|^{n+1}}{(z-1)^n} \right) \left(\frac{n+1}{n+2} \right) = |z-1|$

So for $|z-1| < 1$, this series converges, the series S is then absolutely convergent $\Rightarrow$ it is convergent. So radius of convergence $r \geq 1$.

But for $z = 2$ $S = \sum_{n=0}^{\infty} \frac{1}{n+1}$ which is the harmonic series and diverges.

So the radius of convergence is 1.



2. Find the real and imaginary parts of the principal value of the following.

2.a) $\tan(2i)$ (5 points)

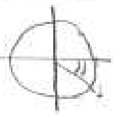
$$\tan 2i = \frac{\sin 2i}{\cos 2i} = \frac{\dfrac{e^{i(2i)} - e^{-i(2i)}}{2i}}{\dfrac{e^{i(2i)} + e^{-i(2i)}}{2}} = \frac{e^{-2} - e^{2}}{i(e^{-2} + e^{2})}$$

$$= i\left[\frac{e^{2} - e^{-2}}{e^{2} + e^{-2}}\right]$$

$$\operatorname{Re}(\tan 2i) = 0$$

$$\operatorname{Im}(\tan 2i) = \frac{e^{2} - e^{-2}}{e^{2} + e^{-2}}$$

2.b) $\ln(1-i)$ (5 points)



$$|1-i| = \sqrt{2}, \quad \theta = \tan^{-1}(-1) = -\frac{\pi}{4}, \ \frac{7\pi}{4}$$

$$\Rightarrow \ 1-i = \sqrt{2}\, e^{i\frac{7\pi}{4}} = e^{\ln\sqrt{2} + i\frac{7\pi}{4} + 2\pi i k} \qquad k \in \mathbb{Z}$$

$$\Rightarrow \ \ln(1-i) = \ln\sqrt{2} + i\frac{7\pi}{4} + 2\pi i k$$

$$PV\left[\ln(1-i)\right] = \ln\sqrt{2} + i\frac{7\pi}{4} = \underbrace{\frac{1}{2}\ln 2}_{\text{Real part}} + i\underbrace{\left(\frac{7\pi}{4}\right)}_{\text{Imaginary part.}}$$

2.c) $(1-i)^{(1+i)}$ (10 points)

$$(1-i)^{1+i} = e^{(1+i)\ln(1-i)} \qquad k \in \mathbb{Z}$$

$$(1+i)\ln(1-i) = (1+i)\left(\ln\sqrt{2} + i\frac{7\pi}{4} + 2\pi i k\right)$$

$$= \left(\ln\sqrt{2} - \frac{7\pi}{4} - 2\pi k\right) + i\left(\ln\sqrt{2} + \frac{7\pi}{4} + 2\pi k\right)$$

$$PV\left[(1-i)^{1+i}\right] = e^{\left(\ln\sqrt{2} - \frac{7\pi}{4}\right) + i\left(\ln\sqrt{2} + \frac{7\pi}{4}\right)}$$

$$= e^{\ln\sqrt{2} - \frac{7\pi}{4}}\, e^{i\left(\ln\sqrt{2} + \frac{7\pi}{4}\right)}$$

$$= e^{\ln\sqrt{2} - \frac{7\pi}{4}} \cos\left(\ln\sqrt{2} + \frac{7\pi}{4}\right) + i\, e^{\ln\sqrt{2} + \frac{\pi}{4}} \sin\left(\ln\sqrt{2} + \frac{7\pi}{4}\right)$$

$$= \underbrace{e^{\frac{1}{2}\ln 2 - \frac{7\pi}{4}} \cos\left(\frac{\ln 2}{2} + \frac{7\pi}{4}\right)}_{\text{Real part}} + i\underbrace{\left[e^{\frac{1}{2}\ln 2 - \frac{7\pi}{4}} \sin\left(\frac{\ln 2}{2} + \frac{7\pi}{4}\right)\right]}_{\text{Imaginary part}}$$

3. Let $\mathbb{U}$ be a unitary $n \times n$ matrix, λ be an eigenvalue of $\mathbb{U}$, and $\vec{v}_1, \vec{v}_2 \in \mathbb{C}^n$ be two eigenvectors of $\mathbb{U}$ with different eigenvalues λ_1 and λ_2, respectively. Show that

3.a) $|\lambda| = 1$, (10 points) let $\vec{v}$ be an eigenvector of $\mathbb{U}$ with eigenvalue λ: Because $\mathbb{U}$ is unitary

$$\langle \mathbb{U}\vec{v}, \mathbb{U}\vec{v} \rangle = \langle \vec{v}, \vec{v} \rangle \quad \text{but} \quad \mathbb{U}\vec{v} = \lambda\vec{v} \Rightarrow$$

$$\underbrace{\langle \lambda\vec{v}, \lambda\vec{v} \rangle}_{\lambda^*\lambda \langle \vec{v}, \vec{v} \rangle} = \langle \vec{v}, \vec{v} \rangle$$

$$\lambda^*\lambda \langle \vec{v}, \vec{v} \rangle = \langle \vec{v}, \vec{v} \rangle \Rightarrow (|\lambda|^2 - 1) \langle \vec{v}, \vec{v} \rangle = 0$$

$$\vec{v} \neq \vec{0} \Rightarrow \langle \vec{v}, \vec{v} \rangle \neq 0 \Rightarrow |\lambda|^2 = 1 \Rightarrow \boxed{|\lambda| = 1}$$

3.b) $\vec{v}_1$ and $\vec{v}_2$ are orthogonal. (15 points)

$$\mathbb{U}\vec{v}_1 = \lambda_1\vec{v}_1, \quad \mathbb{U}\vec{v}_2 = \lambda_2\vec{v}_2$$

$$\langle \mathbb{U}\vec{v}_1, \mathbb{U}\vec{v}_2 \rangle = \langle \vec{v}_1, \vec{v}_2 \rangle$$

$$\langle \lambda_1\vec{v}_1, \lambda_2\vec{v}_2 \rangle = \langle \vec{v}_1, \vec{v}_2 \rangle$$

$$\lambda_1^*\lambda_2 \langle \vec{v}_1, \vec{v}_2 \rangle = \langle \vec{v}_1, \vec{v}_2 \rangle \Rightarrow$$

$$(1 - \lambda_1^*\lambda_2) \langle \vec{v}_1, \vec{v}_2 \rangle = 0$$

$|\lambda_1| = |\lambda_2| = 1 \Rightarrow \lambda_1 = e^{i\theta_1}$ for some $\theta_1 \in \mathbb{R}$

$$\Rightarrow \lambda_1^* = e^{-i\theta_1} = \frac{1}{\lambda_1} \Rightarrow$$

$$\left(1 - \frac{\lambda_2}{\lambda_1}\right) \langle \vec{v}_1, \vec{v}_2 \rangle = 0 \qquad \lambda_1 \neq \lambda_2 \Rightarrow 1 - \frac{\lambda_2}{\lambda_1} \neq 0$$

$$\Rightarrow \boxed{\langle \vec{v}_1, \vec{v}_2 \rangle = 0} \quad \blacksquare$$

4. Let $\mathbb{A} = \begin{pmatrix} i & 1+i \\ 1-i & i \end{pmatrix}$.

4.a) Find the eigenvalues of $\mathbb{A}$. (10 points)

$\det \begin{pmatrix} i-\lambda & 1+i \\ 1-i & i-\lambda \end{pmatrix} = 0$ ②

$(i-\lambda)^2 - (1+i)(1-i) = 0 \Rightarrow (\lambda - i)^2 - 2 = 0 \Rightarrow$ ③

$\lambda - i = \pm\sqrt{2} \Rightarrow \boxed{\lambda = i \pm \sqrt{2}}$ ⑤

$\lambda_1 = i - \sqrt{2}$, $\lambda_2 = i + \sqrt{2}$ are the eigenvalues.

4.b) Obtain an orthonormal basis of $\mathbb{C}^2$ that consists of eigenvectors of $\mathbb{A}$. (20 points)

$(\mathbb{A} - \lambda \mathbb{I})\vec{v} = \vec{0} \Rightarrow \begin{bmatrix} i-\lambda & 1+i \\ 1-i & i-\lambda \end{bmatrix} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{v}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

For $\lambda = \lambda_1 = i - \sqrt{2}$ $\begin{bmatrix} i-(i-\sqrt{2}) & 1+i \\ 1-i & i-(i-\sqrt{2}) \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \sqrt{2}\, a_1 + (1+i)\, b_1 = 0 \Rightarrow b_1 = -\frac{\sqrt{2}}{1+i} a_1$

$(1-i)\, a_1 + \sqrt{2}\, b_1 = 0$ ✓

$\vec{v}_1 = \begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{1+i} \end{pmatrix} a_1$ for some $a \in \mathbb{C}$. ⑥

For $\lambda = \lambda_2 = i + \sqrt{2}$ by symmetry $(\sqrt{2} \longleftrightarrow -\sqrt{2})$

$\vec{v}_2 = \begin{pmatrix} 1 \\ \frac{\sqrt{2}}{1+i} \end{pmatrix} a_2$ for some $a' \in \mathbb{C}$ ⑥

$\langle \vec{v}_1, \vec{v}_2 \rangle = \vec{v}_1^{\,*} \vec{v}_2 = \left(1 \quad -\frac{\sqrt{2}}{1-i}\right) \begin{pmatrix} 1 \\ \frac{\sqrt{2}}{1+i} \end{pmatrix} a_1^* a_2$

$= \left[1 - \frac{2}{(1-i)(1+i)}\right] a_1^* a_2 = 0$ ③

so it is sufficient to choose a_1, a_2 so that $\vec{v}_1$ & $\vec{v}_2$ are unit $\Rightarrow$

$1 = \|\vec{v}_1\|^2 = \left[1 + \frac{2}{2}\right] |a_1|^2 \overset{6}{=} 2|a_1|^2$ choose $\Rightarrow a_1 = \frac{1}{\sqrt{2}}$ ②

$1 = \|\vec{v}_2\|^2 = \left[1 + \frac{2}{2}\right] |a_2|^2 = 2|a_2|^2 \Rightarrow a_2 = \frac{1}{\sqrt{2}}$ ②

So the orthonormal basis is

$$\{\hat{v}_1, \hat{v}_2\}$$

where

$$\hat{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{1+i} \end{pmatrix}, \quad \hat{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \frac{\sqrt{2}}{1+i} \end{pmatrix} \quad 1$$

The most general such basis vectors are

$$\hat{v}_1 = \frac{e^{i\alpha_1}}{\sqrt{2}}\begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{1+i} \end{pmatrix}, \quad \hat{v}_2 = \frac{e^{i\alpha_2}}{\sqrt{2}}\begin{pmatrix} 1 \\ \frac{\sqrt{2}}{1+i} \end{pmatrix}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$ are arbitrary
