Solu'ns to Final Exam

Problem 1 (10 points) Solve the following initial-value problem and find the largest range of values of t for which your solution is valid.

$$ty'(t) + ty(t) = 1 - y(t), \qquad y(1) = e.$$

$$ty' + (t+1)y = 1$$

$$y' + (1+\tfrac{1}{t})y = \frac{1}{t}$$

$$\mu y' + \mu(1+\tfrac{1}{t})y = \frac{\mu}{t}$$

$$\frac{d}{dt}(\mu y) - \underbrace{[\mu' - (1+\tfrac{1}{t})\mu]}_{0}y = \frac{\mu}{t}$$

$$\Rightarrow \quad \frac{\mu'}{\mu} = 1 + \frac{1}{t} \Rightarrow \quad \mu = k e^{t + \ln t} = \underset{\overset{\shortparallel}{1}}{k}\, t e^{t}$$

$$\frac{d}{dt}(te^t y) = e^t$$

$$te^t y = e^t + c$$

$$\Rightarrow \quad y(t) = \frac{1}{t} + \frac{ce^{-t}}{t}$$

$$y(1) = e \quad \Rightarrow \quad e = 1 + ce^{-1}$$

$$\Rightarrow \quad c = e(e-1)$$

$$\Rightarrow \quad y(t) = \frac{1}{t} + \frac{(e-1)e^{1-t}}{t}$$

$$\Rightarrow \quad \boxed{y(t) = \frac{1}{t}\left[1 + (e-1)e^{1-t}\right]}$$

Problem 2 (15 points) Find the function $q(t)$ such that $y_1(t) = t^{-1}$ is a solution of the equation

$$y'' - tq(t)y' + q(t)y = 0, \qquad t > 0,$$

and obtain the general solution of this equation for this choice of $q(t)$.

$y_1' = -t^{-2}, \quad y_1'' = 2t^{-3}$

$2t^{-3} - tq(t)(-t^{-2}) + q(t)t^{-1} = 0$

$2t^{-3} + t^{-1}q(t) + t^{-1}q(t) = 0$

$2t^{-1}q(t) = -2t^{-3} \Rightarrow \boxed{q(t) = -t^{-2}}$

$\Rightarrow y'' + t^{-1}y' - t^{-2}y = 0$

$W(y_1, y_2) = c\,e^{-\int \frac{dt}{t}} = c\,e^{-\ln t} = \frac{c}{t}$

For $c = 1 \Rightarrow$

$y_1 y_2' - y_2 y_1' = \frac{1}{t}$

$\frac{1}{t}y_2' - \left(-\frac{1}{t^2}\right)y_2 = \frac{1}{t} \Rightarrow y_2' + \frac{y_2}{t} = 1$

$ty_2' + y_2 = t$

$\frac{d}{dt}(ty_2) = t \Rightarrow ty_2 = \frac{t^2}{2} + c$

$\Rightarrow y_2 = \frac{t}{2} + \frac{c}{t} \Rightarrow y_2(t) = \frac{t}{2}$ is a solution.

$\Rightarrow \boxed{y(t) = \frac{c_1}{t} + \frac{c_2 t}{2}} = \boxed{\frac{c_1}{t} + \tilde{c}_2 t}$ is the general solution.

$c_1, \tilde{c}_2 \in \mathbb{R}$

Problem 3 Let y_1 and y_2 be solutions of the differential equation

$$y'' + p(t)y + q(t) = 0, \qquad t \in (a,b), \qquad (*)$$

where p and q are continuous functions on the interval (a,b).

a (5 points) Show that if y_1 and y_2 are linearly-dependent, then their Wronskian $W(y_1(t), y_2(t))$ is identically zero for all $t \in (a,b)$.

If y_1 or y_2 is identicalls zero, the $W = y_1 y_2' - y_2 y_1' = 0$

If both y_1 & y_2 are nonzero, $\exists$ a real numbr $\alpha \in \mathbb{R}$

such that $y_1 = \alpha y_2 \Rightarrow y_1' = \alpha y_2' \Rightarrow$

$$W = y_1 y_2' - y_2 y_1' = \alpha y_2 y_2' - y_2(\alpha y_2') = 0.$$

b (5 points) Show that if $W(y_1(t_0), y_2(t_0)) = 0$ for some $t_0 \in (a,b)$, then y_1 and y_2 are linearly-dependent.

Should find c_1, c_2 not both zero such that

$$\boxed{c_1 y_1(t) + c_2 y_2(t) = 0, \quad \forall t \in (a,b)} \qquad (I)$$

If either of y_1 or y_2 vanishes, then they are linearly-dependent. So suppose that none of them vanishes $\Rightarrow \exists\, t_* \in (a,b)$, $y_1(t_*) \neq 0$.

By Abel's theorem $W(y_1(t_*), y_2(t_*)) = 0$.

Let $\boxed{\phi(t) := -y_2(t_*) y_1(t) + y_1(t_*) y_2(t)}$. This is a solution of the equation, because it is a linear combination of solutions. $\phi(t_*) = 0$ and

$$\phi'(t_*) = -y_2(t_*) y_1'(t_*) + y_1(t_*) y_2'(t_*) = W(y_1(t_*), y_2(t_*)) = 0$$

$\Rightarrow$ $\phi(t)$ is a solution of $(*)$ satisfy $\phi(t_*) = \phi'(t_*) = 0$. But zero also satisfies the same initial value problem. By uniqueness theorem for solution of this problem we have $\phi(t) = 0$, $\forall t \in (a,b)$

$\Rightarrow$ (I) holds for $c_1 = -y_2(t_*)$ & $c_2 = y_1(t_*) \neq 0$. $\blacksquare$

Problem 4 (10 points) Use the method of Laplace transform to solve the following initial-value problem.

$$y''(t) - y(t) = e^{2t},$$
$$y(0) = 1, \quad y'(0) = 0.$$

Let $Y(s) := \mathcal{L}\{y(t)\}$ $\Rightarrow$

$$s^2 Y(s) - \overset{1}{y(0)}\, s - \overset{0}{y'(0)} - Y(s) = \frac{1}{s-2}$$

$$(s^2-1)\,Y(s) - s = \frac{1}{s-2}$$

$$Y(s) = \frac{s}{s^2-1} + \frac{1}{(s-2)(s^2-1)} \Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s^2-1)}\right\}$$

$$\frac{s}{s^2-1} = \frac{1}{2}\left(\frac{1}{s-1} + \frac{1}{s+1}\right) \Rightarrow \boxed{\mathcal{L}^{-1}\left\{\frac{s}{s^2-1}\right\} = \frac{1}{2}(e^{t} + e^{-t})}$$

$$\frac{1}{(s-2)(s^2-1)} = \frac{A}{s-2} + \frac{B}{s-1} + \frac{C}{s+1}$$

$$= \frac{A(s^2-1) + B(s-2)(s+1) + C(s-2)(s-1)}{(s-2)(s^2-1)}$$

$$\Rightarrow \begin{cases} A + B + C = 0 \\ -B - 3C = 0 \\ -A - 2B + 2C = 1 \end{cases} \Rightarrow B = -3C$$

$A - 2C = 0 \Rightarrow A = 2C$

$\Rightarrow -2C + 6C + 2C = 1 \Rightarrow C = \frac{1}{6}$

$$\Rightarrow A = \frac{1}{3}, \quad B = -\frac{1}{2}, \quad C = \frac{1}{6}$$

$$\Rightarrow \frac{1}{(s-2)(s^2-1)} = \frac{1}{3}\left(\frac{1}{s-2}\right) - \frac{1}{2}\left(\frac{1}{s-1}\right) + \frac{1}{6}\left(\frac{1}{s+1}\right)$$

$$\Rightarrow \mathcal{L}\left\{\frac{1}{(s-2)(s^2-1)}\right\} = \frac{e^{2t}}{3} - \frac{e^{t}}{2} + \frac{e^{-t}}{6}$$

$$\Rightarrow y(t) = \frac{1}{2}(e^{t} + e^{-t}) + \frac{e^{2t}}{3} - \frac{e^{t}}{2} + \frac{e^{-t}}{6}$$

$$\Rightarrow \boxed{y(t) = \frac{2}{3}e^{-t} + \frac{e^{2t}}{3}}$$

Problem 5 (15 points) Find the general solution of the following system of equations.

$$x_1'(t) = 3x_1(t) + x_2(t),$$
$$x_2'(t) = -2x_1(t) + x_2(t).$$

$$\vec{x} := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbb{A} := \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}, \quad (\mathbb{A} - r\mathbb{1})\vec{\xi} = \vec{0}$$

$$\Rightarrow \begin{bmatrix} 3-\lambda & 1 \\ -2 & 1-\lambda \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \det \begin{bmatrix} 3-\lambda & 1 \\ -2 & 1-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (\lambda-3)(\lambda-1)+2 = 0 \Rightarrow \lambda^2 - 4\lambda + 5 = 0$$

$$\Rightarrow \lambda = 2 \pm \sqrt{4-5} = 2 \pm i$$

For $\lambda = \lambda_1 := 2+i$: $\begin{bmatrix} 3-(2+i) & 1 \\ -2 & 1-(2+i) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow (1-i)\xi_1 + \xi_2 = 0 \Rightarrow \xi_2 = (-1+i)\xi_1$$

Take $\xi_1 = 1 \Rightarrow \vec{\xi} = \begin{bmatrix} 1 \\ -1+i \end{bmatrix}$

$\Rightarrow \vec{x}(t) = e^{(2+i)t} \begin{bmatrix} 1 \\ -1+i \end{bmatrix}$ is a complex solution.

$$\vec{x}^{(1)}(t) = \mathrm{Re}(\vec{x}(t)) = \frac{1}{2}\left(e^{(2+i)t} \begin{bmatrix} 1 \\ -1+i \end{bmatrix} + e^{(2-i)t} \begin{bmatrix} 1 \\ -1-i \end{bmatrix} \right)$$

$$= e^{2t} \begin{bmatrix} \cos t \\ -\cos t - \sin t \end{bmatrix}$$

$$\vec{x}^{(2)}(t) = \mathrm{Im}(\vec{x}(t)) = \frac{1}{2i}\left(e^{(2+i)t} \begin{bmatrix} 1 \\ -1+i \end{bmatrix} - e^{(2-i)t} \begin{bmatrix} 1 \\ -1-i \end{bmatrix} \right)$$

$$= e^{2t} \begin{bmatrix} \sin t \\ -\sin t + \cos t \end{bmatrix}$$

$$\vec{x}(t) = c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t) = c_1 e^{2t} \begin{bmatrix} \cos t \\ -\cos t - \sin t \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} \sin t \\ -\sin t + \cos t \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} x_1(t) &= c_1 e^{2t} \cos t + c_2 e^{2t} \sin t \\ x_2(t) &= -c_1 e^{2t}(\sin t + \cos t) + c_2 e^{2t}(-\sin t + \cos t) \end{aligned}}$$

Problem 6 (15 points) Let $\mathbf{P}(t)$ be a 2×2 matrix whose entries are continuous functions of t in $\mathbb{R}$, $\mathbf{\Psi}(t) := \begin{bmatrix} t & 1 \\ -1 & t \end{bmatrix}$, and $\mathbf{g}(t) := \begin{bmatrix} t^2+1 \\ -t^2+1 \end{bmatrix}$. Given that $\mathbf{\Psi}(t)$ is a fundamental matrix for the homogeneous system of linear differential equations: $\mathbf{X}'(t) = \mathbf{P}(t)\mathbf{X}(t)$, solve the following initial value problem.

$$\mathbf{X}'(t) = \mathbf{P}(t)\mathbf{X}(t) + \mathbf{g}(t),$$

$$\mathbf{X}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

a) $\Psi' = P\Psi$

$\Downarrow$

$P = \Psi'\Psi^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\Psi^{-1}$

$= \Psi^{-1} = \frac{1}{t^2+1}\begin{bmatrix} t & -1 \\ 1 & t \end{bmatrix}$

b) $\bar{X}(t) = \Psi(t)\vec{c} + \Psi(t)\int_0^t \Psi(s)^{-1}\bar{g}(s)\,ds$

$\Psi(t)^{-1} = \frac{1}{t^2+1}\begin{bmatrix} t & -1 \\ 1 & t \end{bmatrix}$

$\Psi(s)^{-1} g(s) = \frac{1}{s^2+1}\begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}\begin{bmatrix} s^2+1 \\ -s^2-1 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} s+1 \\ 1-s \end{bmatrix}$

$\int_0^t \Psi(s)^{-1} g(s)\,ds = \begin{bmatrix} (\frac{s^2}{2}+s)\big|_0^t \\ (s-\frac{s^2}{2})\big|_0^t \end{bmatrix} = \begin{bmatrix} \frac{t^2}{2}+t \\ -\frac{t^2}{2}+t \end{bmatrix}$

$\Psi(t)\int_0^t \Psi(s)^{-1} g(s)\,ds = \begin{bmatrix} t & 1 \\ -1 & t \end{bmatrix}\begin{bmatrix} \frac{t^2}{2}+t \\ -\frac{t^2}{2}+t \end{bmatrix}$

$= \begin{bmatrix} \frac{t^3}{2}+t^2-\frac{t^2}{2}+t \\ -\frac{t^2}{2}-t-\frac{t^3}{2}+t^2 \end{bmatrix} = \begin{bmatrix} \frac{t^3}{2}+\frac{t^2}{2}+t \\ -\frac{t^3}{2}+\frac{t^2}{2}-t \end{bmatrix}$

$\bar{X}(0) = \Psi(0)\vec{c} \Rightarrow \vec{c} = \Psi(0)^{-1}\bar{X}(0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\Psi(t)\vec{c} = \begin{bmatrix} t & 1 \\ -1 & t \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -t+1 \\ 1+t \end{bmatrix}$

$\Rightarrow \bar{X}(t) = \begin{bmatrix} 1-t \\ 1+t \end{bmatrix} + \begin{bmatrix} \frac{t^3}{2}+\frac{t^2}{2}+t \\ -\frac{t^3}{2}+\frac{t^2}{2}-t \end{bmatrix}$

$\Rightarrow \boxed{\bar{X}(t) = \begin{bmatrix} \frac{t^3}{2}+\frac{t^2}{2}+1 \\ -\frac{t^3}{2}+\frac{t^2}{2}+1 \end{bmatrix}}$

Problem 7 Let $f : \mathbb{R} \to \mathbb{R}$ be a period function with period 2π, i.e., $f(x + 2\pi) = f(x)$, and $f(x) = x$ for $-\pi < x \leq \pi$.

7a (7 points) Find the Fourier series for $f(x)$. $\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right]$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x\cos(nx)}_{\text{odd}}\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(nx)}{n^2} - \frac{x\cos(nx)}{n}\right]\Bigg|_{-\pi}^{\pi}$$

$$= \frac{2}{\pi}\left(-\frac{\pi\cos(n\pi)}{n}\right) = -\frac{2}{n}\cos(n\pi)$$

$$= \frac{2}{n}(-1)^{n+1}$$

$$\int x\sin(\alpha x)\,dx = -\frac{\partial}{\partial\alpha}\int \cos(\alpha x)\,dx$$

$$= -\frac{\partial}{\partial\alpha}\frac{\sin(\alpha x)}{\alpha} + c$$

$$= +\frac{\sin(\alpha x)}{\alpha^2} - \frac{x\cos(\alpha x)}{\alpha} + c$$

Fourier series for $f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}\sin(nx)}{n}$

7b (4 points) Is the Fourier series for $f(x)$ converges to $f(x)$ for $x = \pi$? Why?

$f(\pi) = \pi$, $\quad \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}\sin(\pi n)}{n} = \sum_{n=1}^{\infty} 0 = 0$

So the Fourier series for $f(x)$ does not converge to $f(x)$ for $x = \pi$.

7c (4 points) Is the Fourier series for $f(x)$ converges to $f(x)$ for $x = \frac{\pi}{2}$? Why?

$f(x)$ is continuous at $x = \frac{\pi}{2}$ so the Fourier series converges to $f(x)$ at $x = \frac{\pi}{2}$, i.e., it converges to $f(\frac{\pi}{2}) = \frac{\pi}{2}$ (By Fourier Convergence theorem.)

Problem 8 (20 points) Solve the following problem.

$$u_t = 4u_{xx}, \quad x \in (0,\pi), \quad t > 0$$
$$u(0,t) = 1, \quad u(\pi,t) = 1+\pi, \quad t > 0$$
$$u(x,0) = 3\sin(2x) + \sin(5x) + x + 1, \quad x \in (0,\pi).$$

$w = u + ax + b$

$w(0,t) = 1 + b = 0 \Rightarrow \quad b = -1$

$w(\pi,t) = 1 + \pi + a\pi - 1 = 0 \Rightarrow \quad a = -1$

$\Rightarrow \quad \boxed{w(x,t) = u(x,t) - x - 1}$

$\Rightarrow \quad w(x,0) = 3\sin(2x) + \sin(5x)$

$$\Rightarrow \begin{cases} w_t = 4w_{xx} \\ w(0,t) = 0, \quad w(\pi,t) = 0 \\ w(x,0) = 3\sin(2x) + \sin(5x) \end{cases}$$

$w(x,t) = X(x)T(t) \Rightarrow \quad XT' = 4X''T$

$\Rightarrow \quad \dfrac{X''}{X} = \dfrac{1}{4}\dfrac{T'}{T} = \lambda \Rightarrow \quad T = c_1 e^{4\lambda t}$

$$\begin{matrix} X'' = \lambda X \\ X(0) = X(\pi) = 0 \end{matrix} \quad \Rightarrow \begin{cases} X = a_n \sin(nx) \\ \lambda = -n^2 \end{cases} \quad n = 1, 2, \ldots$$

$\Rightarrow \quad w(x,t) = \displaystyle\sum_{n=1}^{\infty} b_n e^{-4n^2 t} \sin(nx)$

$w(x,0) = 3\sin(2x) + \sin(5x)$

$b_2 = 3, \quad b_5 = 1, \quad b_n = 0$ for $n \neq 3, 5$

$w(x,t) = 3e^{-16t}\sin(2t) + e^{-100t}\sin(5t)$

$\Rightarrow$

$\Rightarrow \quad \boxed{u(x,t) = x + 1 + 3e^{-16t}\sin(2t) + e^{-100t}\sin(5t)}$