Math 204, Fall 2016
Midterm Exam 2

Problem 1 Suppose that the following initial-value problem has a power series solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$.

$$y''(x) + \int_0^x y(s)ds = x,$$
$$y(0) = 0, \quad y'(0) = 1.$$

1a (10 points) Find the recurrence relation satisfied by a_n.

$$Y''(X) = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$
$$= \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m$$

$$\int_0^x Y(s)\,ds = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = \sum_{m=1}^{\infty} \frac{a_{m-1}}{m} x^m$$

$$\Rightarrow \sum_{m=0}^{\infty} (m+1)(m+2)a_{m+2} x^m + \sum_{m=1}^{\infty} \frac{a_{m-1}}{m} x^m = x$$

$$\Rightarrow 2a_2 + 6a_3 x + \sum_{m=2}^{\infty} (m+1)(m+2)a_{m+2} x^m +$$
$$a_0 x + \sum_{m=2}^{\infty} \frac{a_{m-1}}{m} x^m = x$$

$$\Rightarrow 2a_2 + (6a_3 + a_0 - 1)x + \sum_{m=2}^{\infty} \left[(m+1)(m+2)a_{m+2} + \frac{a_{m-1}}{m}\right]x^m = 0$$

$$\Rightarrow \boxed{a_2 = 0}, \quad \& \quad \boxed{a_{m+2} = -\frac{a_{m-1}}{m(m+1)(m+2)} \text{ for } m \geq 2}$$

$$6a_3 + a_0 - 1 = 0,$$
$$\Downarrow$$
$$\boxed{a_3 = \frac{1-a_0}{6}} \rightarrow \boxed{a_3 = \frac{1}{6}}$$

$$Y(0) = 0 \Rightarrow \boxed{a_0 = 0}$$
$$Y'(0) = 1 \Rightarrow \boxed{a_1 = 1}$$

1b (5 points) Determine the first five nonzero terms of the series $\sum_{n=0}^{\infty} a_n x^n$.

$a_0 = 0,\ a_1 = 1,\ a_2 = 0,\ a_3 = \frac{1}{6}$

for $m = 2$: $a_4 = -\frac{a_1}{4!} = -\frac{1}{4!}$

" $m = 3$: $a_5 = -\frac{a_2}{(3)(4)(5)} = 0$

" $m = 4$: $a_6 = -\frac{a_3}{(4)(5)(6)} = -\frac{1}{6!}$

" $m = 5$: $a_7 = -\frac{a_4}{(5)(6)(7)} = +\frac{1}{7!}$

$\vdots$

$$\sum_{n=0}^{\infty} a_n x^n = x + \frac{x^3}{3!} - \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$$

1c (5 points) Solve the recurrence relation you find in part a, i.e., express a_n in terms of n.

Because $a_2 = 0$, we have $a_5 = a_8 = \cdots = a_{3k+2} = \cdots = 0$

$\Rightarrow \boxed{a_{3k+2} = 0 \quad \text{for} \quad k \geq 0}$

$a_1 = 1,\ a_4 = -\frac{1}{4!},\ a_7 = \frac{1}{7!},\ a_{10} = -\frac{1}{10!};\ \cdots$

$$\boxed{a_{3k+1} = \frac{(-1)^k}{(3k+1)!} \quad \text{for} \quad k \geq 0}$$

$a_3 = \frac{1}{3!},\ a_6 = -\frac{1}{6!},\ a_9 = \frac{1}{9!},\ \cdots$

$$\boxed{a_{3k} = \frac{(-1)^{k+1}}{(3k)!} \quad \text{for} \quad k \geq 1}$$

$$a_n = \begin{cases} 0 & \text{if} \quad n = 0 \\ \frac{(-1)^{k+1}}{(3k)!} & \text{if} \quad n = 3k \quad \text{for} \quad k \in \mathbb{Z}^+ \\ \frac{(-1)^k}{(3k+1)!} & \text{if} \quad n = 3k+1 \quad \text{for} \quad k \in \mathbb{Z}^+ \cup \{0\} \\ 0 & \text{if} \quad n = 3k+2 \quad \text{for} \quad k \in \mathbb{Z}^+ \cup \{0\} \end{cases}$$

Problem 2 Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(t) = e^t$ for $t \geq 2$.

2a (6 points) Show that $f(t)$ has a Laplace transform $F(s)$ for $s > 1$.

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \underbrace{\int_0^2 e^{-st} f(t)\,dt}_{I_1(s)} + \underbrace{\int_2^\infty e^{-st} e^t\,dt}_{I_2(s)}$$

f is differentiable $\Rightarrow$ f is continuous $\Rightarrow$ $I_1(s)$ exists, i.e.,

$$I_2(s) = \lim_{N\to\infty} \int_2^N e^{-(s-1)t}\,dt = \lim_{N\to\infty} \left[\frac{e^{-(s-1)N} - e^{-2(s-1)}}{-(s-1)}\right]$$

Becaus $s > 1$ $\lim_{N\to\infty} \frac{e^{-(s-1)N}}{-(s-1)} = 0$ $\Rightarrow$ $I_2(s) = \frac{e^{-2(s-1)}}{s-1}$

$\Rightarrow$ $F(s)$ exists.

2b (4 points) Show that $f'(t)$ has a Laplace transform for $s > 1$.

For $t \geq 2$, $f(t) = e^t$ $\Rightarrow$ $f'(t) = e^t$ for $t \geq 2$

f' is continuous on $\mathbb{R}$ so we can use the same argument we used in part 2a to show that $\int_0^2 e^{-st} f'(t)\,dt$ exists and

$$\int_2^\infty e^{-st} f'(t)\,dt = \frac{e^{-2(s-1)}}{s-1} \quad \text{for } s > 1$$

$\Downarrow$

$\mathcal{L}\{f'(t)\}$ exists for $s > 1$.

2c (5 points) Use the definition of the Laplace transform to show that the Laplace transform of $f'(t)$ is given by $sF(s) - f(0)$ for $s > 1$, i.e., $\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$.

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \int_0^\infty \left[\frac{d}{dt}\left(e^{-st} f(t)\right) - \frac{d}{dt}\left(e^{-st}\right) f(t)\right] dt$$

$$= e^{-st} f(t)\Big|_0^\infty + \int_0^\infty s\,e^{-st} f(t)\,dt$$

$$= \lim_{N\to\infty} e^{-sN} f(N) - f(0) + s\,\mathcal{L}\{f(t)\}$$

$$= \underbrace{\lim_{N\to\infty} e^{-sN} e^{N}}_{\to 0 \text{ becaus } s>1} + sF(s) - f(0) = sF(s) - f(0)$$

Problem 3 Let f and g be differentiable functions satisfying

$$f'(t) - 2g(t) = u_2(t), \tag{1}$$
$$g'(t) + f(t) = 1, \tag{2}$$
$$f(0) = g(0) = 0, \tag{3}$$

where $u_2(t)$ denotes the step function satisfying $u_2(t) = 0$ for $t < 2$ and $u_2(t) = 1$ for $t \geq 2$. Suppose that $f(t)$ and $g(t)$ have respectively Laplace transforms, $F(s)$ and $G(s)$, for $s > 0$.

3a (10 points) Take the Laplace transform of both sides of Equations (1) and (2) and use the result to determine $F(s)$ and $G(s)$.

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) = sF(s)$$

$$\mathcal{L}\{g'(t)\} = sG(s) - g(0) = sG(s)$$

$$\mathcal{L}\{u_2(t)\} = \int_0^\infty e^{-st} u_2(t)\,dt = \int_2^\infty e^{-st}\,dt = \frac{e^{-2s}}{s} \quad \text{for } s>0$$

$$(1) \Rightarrow sF(s) - 2G(s) = \frac{e^{-2s}}{s}$$

$$(2) \Rightarrow sG(s) + F(s) = \frac{1}{s} \Rightarrow F(s) + sG(s) = \frac{1}{s}$$

$$\Updownarrow$$

$$\begin{bmatrix} s & -2 \\ 1 & s \end{bmatrix} \begin{bmatrix} F(s) \\ G(s) \end{bmatrix} = \begin{bmatrix} \frac{e^{-2s}}{s} \\ \frac{1}{s} \end{bmatrix}$$

$$\Rightarrow \begin{cases} \boxed{F(s)} = \dfrac{\det \begin{bmatrix} \frac{e^{-2s}}{s} & -2 \\ \frac{1}{s} & s \end{bmatrix}}{\det \begin{bmatrix} s & -2 \\ 1 & s \end{bmatrix}} = \dfrac{e^{-2s} + \frac{2}{s}}{s^2+2} = \boxed{\dfrac{e^{-2s}}{s^2+2} + \dfrac{2}{s(s^2+2)}} \\[2em] \boxed{G(s)} = \dfrac{\det \begin{bmatrix} s & \frac{e^{-2s}}{s} \\ 1 & \frac{1}{s} \end{bmatrix}}{s^2+2} = \dfrac{1 - \frac{e^{-2s}}{s}}{s^2+2} = \boxed{\dfrac{1}{s^2+2} - \dfrac{e^{-2s}}{s(s^2+2)}} \end{cases}$$

3b (15 points) Use your response to Problem **3a** to obtain $f(t)$ and $g(t)$.

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{ \frac{e^{-2s}}{s^2+2} + \frac{2}{s(s^2+2)} \right\} = \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+2)}\right\}$$

$$\mathcal{L}^{-1}\left\{ \frac{e^{-2s}}{s^2+2} \right\} = \mathcal{L}^{-1}\left\{ \underbrace{\frac{e^{-2s}}{s}}_{=\mathcal{L}\{u_2(t)\}} \cdot \underbrace{\frac{s}{s^2+2}}_{=\mathcal{L}\{\cos\sqrt{2}t\}} \right\} = u_2(t) * \cos(\sqrt{2}t) = \cos(\sqrt{2}t) * u_2(t)$$

$$= \int_0^t \cos(\sqrt{2}(t-\tau))\, u_2(\tau)\, d\tau = \begin{cases} 0 & \text{if } t<2 \\ \int_2^t \cos(\sqrt{2}(t-\tau))\, d\tau & \text{if } t\geq 2 \end{cases}$$

$$= u_2(t)\, \frac{\sin[\sqrt{2}(\tau-t)]}{\sqrt{2}}\Bigg|_{\tau=2}^{\tau=t} = u_2(t)\, \frac{\sin[\sqrt{2}(t-2)]}{\sqrt{2}}$$

$$\mathcal{L}^{-1}\left\{ \frac{1}{s(s^2+2)} \right\} = \mathcal{L}^{-1}\left\{ \underbrace{\frac{1}{s}}_{=\mathcal{L}(1)} \cdot \underbrace{\frac{1}{s^2+2}}_{=\mathcal{L}\left\{\frac{\sin(\sqrt{2}t)}{\sqrt{2}}\right\}} \right\} = 1 * \frac{\sin(\sqrt{2}t)}{\sqrt{2}}$$

$$= \int_0^t \frac{\sin(\sqrt{2}\tau)}{\sqrt{2}}\, d\tau = -\frac{\cos(\sqrt{2}\tau)}{2}\Bigg|_0^t = \frac{1-\cos(\sqrt{2}t)}{2}$$

$$\Rightarrow \boxed{f(t) = \frac{1}{\sqrt{2}}\, u_2(t)\, \sin[\sqrt{2}(t-2)] + 1 - \cos(\sqrt{2}t)}$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{ \frac{1}{s^2+2} - \frac{e^{-2s}}{s(s^2+2)} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{s^2+2} \right\} - \mathcal{L}^{-1}\left\{ \frac{e^{-2s}}{s(s^2+2)} \right\}$$

$$\mathcal{L}^{-1}\left\{ \frac{1}{s^2+2} \right\} = \frac{1}{\sqrt{2}} \sin(\sqrt{2}t)$$

$$\mathcal{L}^{-1}\left\{ \frac{e^{-2s}}{s(s^2+2)} \right\} = \mathcal{L}^{-1}\left\{ \underbrace{\frac{1}{s^2+2}}_{=\mathcal{L}\left\{\frac{\sin(\sqrt{2}t)}{\sqrt{2}}\right\}} \cdot \underbrace{\frac{e^{-2s}}{s}}_{=\mathcal{L}\{u_2(t)\}} \right\} = \frac{\sin(\sqrt{2}t)}{\sqrt{2}} * u_2(t)$$

$$= \int_0^t \frac{\sin[\sqrt{2}(t-\tau)]}{\sqrt{2}}\, u_2(\tau)\, d\tau = \begin{cases} 0 & \text{if } t<2 \\ \int_2^t \frac{\sin(\sqrt{2}(t-\tau))}{\sqrt{2}}\, d\tau & \text{if } t\geq 2 \end{cases}$$

$$= u_2(t)\, \frac{\cos(\sqrt{2}(\tau-t))}{2}\Bigg|_{\tau=2}^{\tau=t}$$

$$= \frac{1}{2} u_2(t) \left(1 - \cos[\sqrt{2}(t-2)]\right)$$

$$\Rightarrow \boxed{g(t) = \frac{1}{\sqrt{2}} \sin(\sqrt{2}t) - \frac{1}{2} u_2(t) \{1 - \cos[\sqrt{2}(t-2)]\}}$$

Problem 4 (15 points) Find the general solution of the following system of equations.

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \mathbf{x}(t).$$

$$\det \begin{bmatrix} 1-r & 0 & -4 \\ 0 & 1-r & 0 \\ 1 & 0 & 1-r \end{bmatrix} = 0 \Rightarrow (1-r)^3 + 4(1-r) = 0$$

$$\Rightarrow (1-r)\left[(r-1)^2+4\right] = 0 \Rightarrow \begin{cases} r = 1 \\ r = 1 \pm 2i \end{cases}$$

$\underline{r = 1:}$ $\begin{bmatrix} 0 & 0 & -4 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow a = c = 0 \Rightarrow \vec{\xi}^{(1)} = b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\underline{r = \rho := 1+2i:}$ $\begin{bmatrix} 1-(1+2i) & 0 & -4 \\ 0 & 1-(1+2i) & 0 \\ 1 & 0 & 1-(1+2i) \end{bmatrix}\begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{aligned} -2i\,d - 4f &= 0 \\ e &= 0 \\ d - 2i f &= 0 \end{aligned} \Rightarrow d = 2if \Rightarrow \vec{\xi}^{(2)} = f\begin{bmatrix} 2i \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2i \\ 0 \\ 1 \end{bmatrix}$$

$\underline{r = \bar{\rho} = 1-2i:}$ $\vec{\xi}^{(3)} = \overline{\vec{\xi}^{(2)}} = \begin{bmatrix} -2i \\ 0 \\ 1 \end{bmatrix}$

$$\boxed{\vec{X}^{(1)}(t) = e^{t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

$\vec{Y}(t) = e^{(1+2i)t}\begin{bmatrix} 2i \\ 0 \\ 1 \end{bmatrix}$ is a complex sol.

So is $\overline{\vec{Y}}(t) = e^{(1-2i)t}\begin{bmatrix} -2i \\ 0 \\ 1 \end{bmatrix}$

$$\boxed{\vec{X}^{(2)} = \frac{1}{2}\left[\vec{Y}(t) + \overline{\vec{Y}}(t)\right] = \frac{1}{2}e^{t}\begin{bmatrix} 2ie^{2it} - 2ie^{-2it} \\ 0 \\ e^{2it} + e^{-2it} \end{bmatrix} = e^{t}\begin{bmatrix} -2\sin(2t) \\ 0 \\ \cos(2t) \end{bmatrix}}$$

$$\boxed{\vec{X}^{(3)} = \frac{1}{2i}\left[\vec{Y}(t) - \overline{\vec{Y}}(t)\right] = \frac{1}{2i}e^{t}\begin{bmatrix} 2ie^{2it} + 2ie^{-2it} \\ 0 \\ e^{2it} - e^{-2it} \end{bmatrix} = e^{t}\begin{bmatrix} 2\cos(2t) \\ 0 \\ \sin(2t) \end{bmatrix}}$$

$$\vec{X}(t) = c_1 e^{t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} -2\sin(2t) \\ 0 \\ \cos(2t) \end{bmatrix} + c_3 e^{t}\begin{bmatrix} 2\cos(2t) \\ 0 \\ \sin(2t) \end{bmatrix}$$

$$\Rightarrow \boxed{\vec{X}(t) = e^{t}\begin{bmatrix} -2c_2\sin(2t) + 2c_3\cos(2t) \\ c_1 \\ c_2\cos(2t) + c_3\sin(2t) \end{bmatrix}}$$

Problem 5 Let $\mathbf{A} = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$.

5a (10 points) Find an explicit expression for the fundamental matrix $e^{t\mathbf{A}}$ of the system $\mathbf{x}'(t) = \mathbf{A}\,\mathbf{x}(t)$, i.e., compute the exponential of $t\mathbf{A}$.

$$\det \begin{bmatrix} -r & 3 \\ -3 & -r \end{bmatrix} = 0 \Rightarrow r^2 + 9 = 0 \Rightarrow r = \pm 3i$$

$$r = \rho = 3i \Rightarrow \underbrace{\begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{\xi}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -3ia + 3b = 0$$

$$\Downarrow$$

$$\boxed{b = ia}$$

$$\Rightarrow \vec{\xi} = a\begin{bmatrix} 1 \\ i \end{bmatrix} \quad \text{set } a = 0 \Rightarrow \boxed{\vec{\xi} = \begin{bmatrix} 1 \\ i \end{bmatrix}}$$

$$r = \bar{\rho} = -3i : \quad \boxed{\vec{\bar{\xi}} = \begin{bmatrix} 1 \\ -i \end{bmatrix}} \text{ is an eigenvector for } \bar{\rho}.$$

$$\text{Let } \mathbb{U} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \Rightarrow \mathbb{U}^{-1} = \frac{1}{-2i}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

$$\Rightarrow \mathbb{U}^{-1}\mathbb{A}\mathbb{U} = \begin{bmatrix} 3i & 0 \\ 0 & -3i \end{bmatrix} =: \mathbb{A}_d$$

$$\Downarrow$$

$$\mathbb{A} = \mathbb{U}\mathbb{A}_d\mathbb{U}^{-1} \Rightarrow e^{t\mathbb{A}} = \mathbb{U}e^{t\mathbb{A}_d}\mathbb{U}^{-1}$$

$$\Rightarrow e^{t\mathbb{A}} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} e^{3it} & 0 \\ 0 & e^{-3it} \end{bmatrix}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e^{3it} & e^{-3it} \\ ie^{3it} & -ie^{-3it} \end{bmatrix}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e^{3it} + e^{-3it} & -ie^{3it} + ie^{-3it} \\ ie^{3it} - ie^{-3it} & e^{3it} + e^{-3it} \end{bmatrix}$$

$$= \begin{bmatrix} \cos(3t) & \frac{-i}{2}[2i\sin(3t)] \\ \frac{i}{2}[2i\sin(3t)] & \cos(3t) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(3t) & \sin(3t) \\ -\sin(3t) & \cos(3t) \end{bmatrix}$$

5b (15 points) Use your response to Problem **5a** to solve the initial value problem:

$$x_1'(t) = 3x_2(t) + 2$$
$$x_2'(t) = -3x_1(t) - 1$$
$$x_1(\pi) = 1, \quad x_2(\pi) = -1.$$

$$\vec{X} := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\vec{X}' = \underbrace{\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}}_{\mathbb{A}} \vec{X} + \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{g(t)} \qquad \Psi(t) = e^{t\mathbb{A}} = \begin{bmatrix} \cos(3t) & \sin(3t) \\ -\sin(3t) & \cos(3t) \end{bmatrix}$$

$$\vec{X}(t) = \Psi(t)\,\vec{c} + \int_{\pi}^{t} \Psi(t)\,\Psi^{-1}(s)\,\vec{g}(s)\,ds$$

$$= e^{t\mathbb{A}}\,\vec{c} + \underbrace{\int_{\pi}^{t} e^{(t-s)\mathbb{A}} \begin{bmatrix} 2 \\ -1 \end{bmatrix} ds}_{\vec{Y}(t)}$$

$$\vec{X}(\pi) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{Y}(t) = \int_{\pi}^{t} \begin{bmatrix} \cos 3(t-s) & \sin 3(t-s) \\ -\sin 3(t-s) & \cos 3(t-s) \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} ds$$

$$= \int_{\pi}^{t} \begin{bmatrix} 2\cos[3(s-t)] + \sin[3(s-t)] \\ 2\sin[3(s-t)] - \cos[3(s-t)] \end{bmatrix} ds$$

$$= \begin{bmatrix} \frac{2}{3}\sin[3(s-t)] - \frac{1}{3}\cos[3(s-t)] \\ -\frac{2}{3}\cos[3(s-t)] - \frac{1}{3}\sin[3(s-t)] \end{bmatrix} \Bigg|_{s=\pi}^{s=t}$$

$$= \frac{1}{3} \begin{bmatrix} -1 - [2\sin(3\pi - 3t) - \cos(3\pi - 3t)] \\ -2 - [-2\cos(3\pi - 3t) - \sin(3\pi - 3t) \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} -1 - 2\sin(3t) - \cos(3t) \\ -2 - 2\cos(3t) + \sin(3t) \end{bmatrix}$$

$$\vec{Y}(\pi) = \vec{0} \Rightarrow e^{\pi\mathbb{A}}\,\vec{c} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \vec{c} = e^{-\pi\mathbb{A}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow e^{t\mathbb{A}}\,\vec{c} = e^{(t-\pi)\mathbb{A}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \cos[3(t-\pi)] & \sin[3(t-\pi)] \\ -\sin[3(t-\pi)] & \cos[3(t-\pi)] \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -\cos 3t & -\sin 3t \\ \sin 3t & -\cos 3t \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\cos 3t + \sin 3t \\ \sin 3t + \cos 3t \end{bmatrix}$$

$$\Downarrow$$

$$\vec{X}(t) = \begin{bmatrix} -\operatorname{Cn}(3t) + \sin(3t) + \frac{1}{3}\left[-1 - 2\sin(3t) - \operatorname{Cn}(3t)\right] \\ \sin(3t) + \operatorname{Cn}(3t) + \frac{1}{3}\left[-2 - 2\operatorname{Cn}(3t) + \sin(3t)\right] \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -1 + \sin(3t) - 4\operatorname{Cn}(3t) \\ -2 + 4\sin(3t) + \operatorname{Cn}(3t) \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} x_1(t) &= \frac{1}{3}\left[\sin(3t) - 4\operatorname{Cn}(3t) - 1\right] \\ x_2(t) &= \frac{1}{3}\left[4\sin(3t) + \operatorname{Cn}(3t) - 2\right] \end{aligned}}$$