Solutions to Math 204
Midterm Exam 1 (Fall 2016)

Problem 1 (15 points) Find the general solution of the following Bernoulli equation.

$$y' + 2y = y^3.$$

$$\Rightarrow \quad \frac{y'}{y^3} + \frac{2}{y^2} = 1$$

let $u := \frac{1}{y^2} \quad \Rightarrow \quad u' = -\frac{2y'}{y^3}$

$$\Rightarrow \quad -\frac{u'}{2} + 2u = 1$$

$$\Rightarrow \quad u' - 4u = -2$$

$$\Rightarrow \quad e^{4t}(e^{-4t}u)' = -2$$

$$\Rightarrow \quad e^{-4t}u = -2\int e^{-4t}\,du = -2\left[\frac{e^{-4t}}{-4} + c\right]$$

$$\Rightarrow \quad u = e^{4t}\left(-\frac{e^{-4t}}{2} - 2c\right)$$

$$\Rightarrow \quad u = -\frac{1}{2} - 2ce^{4t}$$

let $a := -2c \quad \Rightarrow \quad u = ae^{4t} - \frac{1}{2}$

$$\Rightarrow \quad y = \pm\frac{1}{\sqrt{u}}$$

$$\Rightarrow \quad \boxed{y = \pm\frac{1}{\sqrt{ae^{4t} - \frac{1}{2}}}}$$

Problem 2a (5 points) Find a real number α so that the following differential equation is exact.

$$\underbrace{\alpha x e^{2xy}}_{N} y' + \underbrace{y e^{2xy} + 3x}_{M} = 0,$$

$$\frac{\partial M}{\partial y} = e^{2xy} + 2xy\, e^{2xy}$$

$$\frac{\partial N}{\partial x} = \alpha e^{2xy} + 2\alpha\, xy\, e^{2xy}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \boxed{\alpha = 1}$$

Problem 2b (15 points) Find the implicit solution of the above equation for the value of α you find in Problem 2a.

$$M = \frac{\partial F}{\partial x}, \qquad N = \frac{\partial F}{\partial y}$$

$$\Downarrow$$

$$F = \int M\,dx = \int (y e^{2xy} + 3x)\,dx = \frac{y e^{2xy}}{2y} + \frac{3x^2}{2} + g(y)$$

$$= \frac{1}{2} e^{2xy} + \frac{3}{2} x^2 + g(y)$$

$$\frac{\partial F}{\partial y} = x e^{2xy} + g' = N = x e^{2xy} \Rightarrow g' = 0$$

$$\Rightarrow g = \text{const.}$$

$$\Rightarrow F = \frac{1}{2} e^{2xy} + \frac{3}{2} x^2 + g$$

Implicit solution: $F = \text{const} \Rightarrow$

$$\boxed{e^{2xy} + 3x^2 = c}$$ when c is a constant.

Problem 3 (20 points) Let y_1 and y_2 be solutions of a second order homogeneous linear differential equation, $y'' + p(x)y' + q(x)y = 0$, in $\mathbb{R}$. Suppose that

$$y_1(x) + y_2(x) = e^{-x}, \qquad W[y_1(x), y_2(x)] = e^{x},$$

where $W[y_1, y_2]$ is the Wronskian of y_1 and y_2.

a) (5 points) Find $p(x)$.

$$W = W[y_1, y_2] = c\,e^{-\int p(x)\,dx} \Rightarrow \frac{W'}{W} = -p(x)$$

But $W = e^{x} \Rightarrow \boxed{p(x) = -1}$

b) (5 points) Find $q(x)$. Becaus sum of any two solutions is also a solution of the above eqn. e^{-x} is a solution $\Rightarrow$

$$(e^{-x})'' + (-1)(e^{-x})' + q(x)e^{-x} = 0$$

$$\Rightarrow e^{-x}(1 + 1 + q(x)) = 0 \Rightarrow \boxed{q(x) = -2}$$

c) (10 points) Find the general form of y_1 and y_2.

$$y_2 = e^{-x} - y_1 \Rightarrow e^{x} = W = y_1(-e^{-x} - y_1') - (e^{-x} - y_1)y_1'$$

$$e^{x} = e^{-x}(-y_1 - y_1') \Rightarrow \underbrace{y_1' + y_1}_{e^{-x}(e^{x}y_1)'} = -e^{2x}$$

$$e^{-x}(e^{x}y_1)' = -e^{2x}$$

$$\Rightarrow e^{x}y_1 = -\int e^{3x}\,dx = -\frac{e^{3x}}{3} + c$$

$$\Rightarrow \boxed{y_1 = -\frac{e^{2x}}{3} + c\,e^{-x}}$$

$$\Rightarrow y_2 = e^{-x} + \frac{e^{2x}}{3} - c\,e^{-x}$$

$$\Rightarrow \boxed{y_2 = \frac{e^{2x}}{3} + (1-c)e^{-x}}$$

when c is an arbitrary constant.

Solution 2

c) **(10 points)** Find the general form of $y_1(t)$ and $y_2(t)$. $p=-1, \quad q=-2 \Rightarrow$

$$\Rightarrow \quad y'' - y' - 2y = 0$$

$$r^2 - r - 2 = 0 \Rightarrow r = \frac{1 \pm \sqrt{1+8}}{2} = \begin{cases} 2 \\ -1 \end{cases}$$

$$\Rightarrow \quad y(t) = c_1 e^{-t} + c_2 e^{2t}$$

$$\Rightarrow \begin{cases} y_1(t) = a_1 e^{-t} + b_1 e^{2t} \\ y_2(t) = a_2 e^{-t} + b_2 e^{2t} \end{cases} \qquad y_1 + y_2 = e^{-t} \Rightarrow \begin{cases} b_1 + b_2 = 0 \\ a_1 + a_2 = 1 \end{cases}$$

$$\Rightarrow \boxed{b_2 = -b_1}, \quad \boxed{a_2 = 1 - a_1} \Rightarrow y_2(t) = (1-a_1)e^{-t} - b_1 e^{2t}$$

$$\Rightarrow W[y_1, y_2] = (a_1 e^{-t} + b_1 e^{2t})\left[(-1+a_1)e^{-t} - 2b_1 e^{2t}\right] - \left[(1-a_1)e^{-t} - b_1 e^{2t}\right]\left[-a_1 e^{-t} + 2b_1 e^{2t}\right] = e^{t}$$

$$\Rightarrow a_1(-1+a_1)e^{-2t} - 2a_1 b_2 e^{t} + b_1(-1+a_1)e^{t} - 2b_1^2 e^{4t} - \left[-a_1(1-a_1)e^{-2t} + 2b_1(1-a_1)e^{t} + a_1 b_1 e^{t} - 2b_1^2 e^{4t}\right] = e^{t}$$

$$\Rightarrow -2a_1 b_2 - b_1 + a_1 b_1 - 2b_1 + 2a_1 b_1 - a_1 b_1 = 1 \Rightarrow \boxed{b_1 = -\frac{1}{3}}$$

$$\Rightarrow \boxed{y_1(t) = a_1 e^{-t} - \frac{e^{2t}}{3}}, \quad \boxed{y_2(t) = (1-a_1)e^{-t} + \frac{e^{2t}}{3}}, \quad a_1 \in \mathbb{R} \text{ is arbitrary}$$

Problem 4 (15 points) Given that $y_1(t) = t^3$ is a solution of the equation:

$$t^2y'' - 4ty' + 6y = 0, \qquad t > 0,$$

find the general solution of this equation.

$$\Rightarrow \; y'' \underbrace{- \frac{4}{t}}_{p(x)} y' + \underbrace{\frac{6}{t^2}}_{q(x)} y = 0$$

$$W[y_1, y_2] = c\, e^{-\int p(t)\,dt}$$

$$t^3 y_2' - 3t^2 y_2 = c\, e^{4\int \frac{dt}{t}} = c\, e^{\ln t^4} = c\, t^4$$

Take $c = 1$

$$\Rightarrow \; y_2' - \frac{3y_2}{t} = t$$

$$(\mu y_2)' - \underbrace{(\mu' + \frac{3}{t}\mu)}_{0} y_2 = \mu t$$

$$\Rightarrow \; \frac{\mu'}{\mu} = -\frac{3}{t} \Rightarrow \ln \mu = -3\ln t + c' \quad (c' \to 0)$$

$$\boxed{\mu(t) = t^{-3}}$$

$$\Rightarrow \; (t^{-3} y_2)' = t^{-3} \cdot t = t^{-2}$$

$$t^{-3} y_2 = -\frac{t^{-1}}{-1} + c'' \quad (c'' \to 0)$$

$$\boxed{y_2 = t^2}$$

$\Rightarrow$ general solution is $\boxed{y(x) = c_1 t^3 + c_2 t^2}$

Problem 5 (10 points) Let $p, q : \mathbb{R} \to \mathbb{R}$ be continuous functions, ϕ_1 be a solution of $y'' + p(x)y' + q(x)y = e^x$ in $\mathbb{R}$, and ϕ_1 be a solution of $y'' + p(x)y' + q(x)y = \sin(3x)$ in $\mathbb{R}$. Obtain a solution of

$$y'' + p(x)y' + q(x)y = 2e^x - \sin(3x)$$

in terms of ϕ_1 and ϕ_2. Justify your response.

$\phi = 2\phi_1 - \phi_2$ is a solution because

$$\begin{aligned}
\phi'' + p(x)\phi' + q(x)\phi &= 2\phi_1'' - \phi_2'' \\
&\quad + p(x)(2\phi_1' - \phi_2') \\
&\quad + q(x)(2\phi_1 - \phi_2) \\
&= 2\left[\phi_1'' + p(x)\phi_1' + q(x)\phi_1\right] + \\
&\quad - \left[\phi_2'' + p(x)\phi_2' + q(x)\phi_2\right] \\
&= 2e^x - \sin(3x)
\end{aligned}$$

Solution 1

Problem 6 (20 points) Solve the following initial-value problem.

$$y'' + y = \frac{1}{\sin t}, \qquad y(\tfrac{\pi}{2}) = y'(\tfrac{\pi}{2}) = 0.$$

$y_H'' + y_H = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow y_H = c_1 \underbrace{\sin t}_{y_1} + c_2 \underbrace{\cos t}_{y_2}$

$y = y_1 u_1 + y_2 u_2 = \sin t\, u_1 + \cos t\, u_2$

$y' = \cos t\, u_1 - \sin t\, u_2$ provided that $\boxed{\sin t\, u_1' + \cos t\, u_2' = 0}$ ①

$\Downarrow$

$y'' = -\sin t\, u_1 - \cos t\, u_2 + \cos t\, u_1' - \sin t\, u_2'$

$\Rightarrow -\sin t\, u_1 - \cos t\, u_2 + \cos t\, u_1' - \sin t\, u_2' + \sin t\, u_1 + \cos t\, u_2 = \frac{1}{\sin t}$

$\Rightarrow \boxed{\cos t\, u_1' - \sin t\, u_2' = \frac{1}{\sin t}}$ ②

① & ② $\Rightarrow$ $u_1' = \dfrac{\det\begin{bmatrix} 0 & \cos t \\ \frac{1}{\sin t} & -\sin t \end{bmatrix}}{\det\begin{bmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{bmatrix}} = \dfrac{-\frac{\cos t}{\sin t}}{-1} = \dfrac{\cos t}{\sin t}$

$\Rightarrow u_1(t) = \int \frac{\cos t}{\sin t}\, dt = \ln|\sin t| + c_1$

$\Rightarrow u_2' = \dfrac{\det\begin{bmatrix} \sin t & 0 \\ \cos t & \frac{1}{\sin t} \end{bmatrix}}{\det\begin{bmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{bmatrix}} = \dfrac{1}{-1} = -1 \Rightarrow u_2(t) = -t + c_2$

$\Rightarrow y(t) = \sin t\,[\ln|\sin t| + c_1] + \cos t\,[-t + c_2]$

$y(t) = c_1 \sin t + c_2 \cos t + \sin t \ln|\sin t| - t\cos t$

For $t \in (0, \pi)$, $\sin t > 0$ &

$y'(t) = c_1 \cos t - c_2 \sin t + \cos t \ln(\sin t) + \cos t - \cos t + t \sin t$

$y(\frac{\pi}{2}) = 0 \Rightarrow c_1 \sin(\frac{\pi}{2}) + c_2 \cos(\frac{\pi}{2}) + \sin\frac{\pi}{2} \ln \underbrace{\sin\frac{\pi}{2}}_{1} - \frac{\pi}{2}\cos(\frac{\pi}{2}) = 0$

$\Rightarrow \boxed{c_1 = 0}$

$y'(\frac{\pi}{2}) = 0 \Rightarrow c_1 \cos(\frac{\pi}{2}) - c_2 \sin(\frac{\pi}{2}) + \cos(\frac{\pi}{2}) \ln(\sin\frac{\pi}{2}) + \frac{\pi}{2}\sin(\frac{\pi}{2}) = 0$

$\Rightarrow \boxed{c_2 = \frac{\pi}{2}}$

$\Rightarrow y(t) = \frac{\pi}{2}\cos t + \sin t \ln|\sin t| - t\cos t$

$\Rightarrow \boxed{y(t) = \sin t \ln|\sin t| - (t - \frac{\pi}{2})\cos t}$

Solution 2

Problem 6 (20 points) Solve the following initial-value problem.

$$y'' + y = \frac{1}{\sin t}, \qquad y(\tfrac{\pi}{2}) = y'(\tfrac{\pi}{2}) = 0.$$

$y_H'' + y_H = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow$

$y_H = c_1 \underbrace{\sin t}_{y_1} + c_2 \underbrace{\cos t}_{y_2}$

$y = y_H + \bar{y}$

$$\bar{y}(t) = \int_{\frac{\pi}{2}}^{t} G(t,s) \frac{1}{\sin s}\, ds \qquad (*)$$

$$G(t,s) = \frac{\det \begin{bmatrix} \sin s & \cos s \\ \sin t & \cos t \end{bmatrix}}{\det \begin{bmatrix} \sin s & \cos s \\ \cos s & -\sin s \end{bmatrix}} = \frac{\cos t \sin s - \sin t \cos s}{-1}$$

$$= \sin t \cos s - \cos t \sin s$$

$$\Rightarrow \bar{y}(t) = \int_{\frac{\pi}{2}}^{t} (\sin t \cos s - \cos t \sin s) \frac{1}{\sin s}\, ds = \sin t \int_{\frac{\pi}{2}}^{t} \frac{\cos s\, ds}{\sin s} - \cos t \int_{\frac{\pi}{2}}^{t} ds$$

$$= \sin t \left[\ln|\sin s|\right]\Big|_{\frac{\pi}{2}}^{t} - \cos t\,(t - \tfrac{\pi}{2})$$

$$= \sin t \ln|\sin t| - (t - \tfrac{\pi}{2}) \cos t$$

$$(*) \Rightarrow \begin{cases} \bar{y}(\frac{\pi}{2}) = 0 \\ \bar{y}'(\frac{\pi}{2}) = \underbrace{G(\frac{\pi}{2}, \frac{\pi}{2})}_{\to 0} \frac{1}{\sin(\frac{\pi}{2})} + \underbrace{\int_{\pi/2}^{\pi/2} \frac{\partial G(t,s)}{\partial t} \frac{1}{\sin s}\, ds}_{\to 0} = 0 \end{cases}$$

$$\underbrace{y(\tfrac{\pi}{2})}_{0} = c_1 \underbrace{\sin(\tfrac{\pi}{2})}_{1} + c_2 \underbrace{\cos(\tfrac{\pi}{2})}_{0} + \underbrace{\bar{y}(\tfrac{\pi}{2})}_{0} \Rightarrow \boxed{c_1 = 0}$$

$$\underbrace{y'(\tfrac{\pi}{2})}_{0} = c_1 \underbrace{\cos(\tfrac{\pi}{2})}_{0} - c_2 \underbrace{\sin(\tfrac{\pi}{2})}_{1} + \underbrace{\bar{y}'(\tfrac{\pi}{2})}_{0} \Rightarrow \boxed{c_2 = 0}$$

$$\Rightarrow \boxed{y(t) = \bar{y}(t) = \sin t \ln|\sin t| - (t - \tfrac{\pi}{2}) \cos t}$$