Solution Sheet

MATH 204: Differential Equations

Midterm 2 - Fall 2014

Duration : 105 minutes

#1	25	
#2	25	
#3	15	
#4	35	
$\sum$	100	

NAME ____________

STUDENT ID ____________

SIGNATURE ____________

- Put your name, student ID and signature in the space provided above.
- No calculators or any other electronic devices are allowed.
- This is a closed-book and closed-notes exam.
- Show all of your work; full credit will not be given for unsupported answers.
- Write your solutions clearly; no credit will be given for unreadable solutions.
- Mark your section below.

SECTION 1 (EMRE MENGI MW 10:00-11:15) ____________

SECTION 2 (ALI ÜLGER, TUTH 14:30-15:45) ____________

SECTION 3 (ALI MOSTAFAZADEH, MW 13:00-14:15) ____________

SECTION 4 (ALTAN ERDOĞAN, TUTH 13:00-14:15) ____________

Problem 1. (25 points) Find the general solution of

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 2y = 0$$

in terms of a power series about 0. Determine the radius of convergence for this general solution.

Suppose the solution is of the form

$$\phi(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\phi'(x) = \sum_{n=\cancel{0}1}^{\infty} n a_n x^{n-1} = \sum_{n=0} (n+1) a_{n+1} x^n$$

$$\phi''(x) = \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^{n-1} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

Plug these in the ODE

$$\frac{d^2\phi}{dx^2} - 2x\frac{d\phi}{dx} - 2\phi = 0$$

$$\left\{\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n\right\} - \underbrace{\left| 2x\left\{\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n\right\}\right|}_{2\sum_{n=1}^{\infty} n a_n x^n} - 2\left\{\sum_{n=0}^{\infty} a_n x^n\right\} = 0$$

This leads to the recurrence

$$(2)(1)a_2 - 2a_0 = 0, \qquad a_2 = a_0$$

$$n \geq 1 \quad (n+2)(n+1)a_{n+2} - 2(n+1)a_n = 0, \qquad a_{n+2} = \frac{2a_n}{(n+2)}$$

Even terms

$$a_2 = a_0$$

$$a_{\cancel{4}4} = \frac{2a_2}{4} = \frac{\cancel{2}a_0}{\cancel{4}\ 2!}$$

$$a_6 = \frac{2a_4}{6} = \frac{\cancel{2\cdot 2}\cdot a_0}{\cancel{6\cdot 4}\ 3!}$$

$$a_8 = \frac{2a_6}{8} = \frac{\cancel{2\cdot 2\cdot 2}\cdot a_0}{\cancel{8\cdot 6\cdot 4}\ 4!}$$

$$a_{2n} = \frac{a_0}{n!}$$

Odd terms

$$a_3 = 2a_1/3$$

$$a_5 = 2a_3/5 = \frac{2\cdot 2 a_1}{5\cdot 3}$$

$$a_7 = 2a_5/7 = \frac{2\cdot 2\cdot 2\, a_1}{7\cdot 5\cdot 3}$$

$$a_{2n+1} = \frac{2^n a_1}{(1)(3)(5)\cdots(2n+1)}$$

Thus the general solution is given by

$$\phi(x) = a_0 \underbrace{\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}}_{\phi_1(x)} + a_1 \underbrace{\sum_{n=0}^{\infty} \frac{2^n x^{2n+1}}{(1)(3)\cdots(2n+1)}}_{\phi_2(x)}$$

for arbitrary $a_0, a_1 \in \mathbb{R}$.

Note: ϕ_1, ϕ_2 are fundamental, since

$$W[\phi_1, \phi_2](0) = \det\left(\begin{bmatrix} \phi_1(0) & \phi_2(0) \\ \phi_1'(0) & \phi_2'(0) \end{bmatrix}\right)$$

$$= 1$$

Radius of convergence

For ϕ_1, by ratio test

$$\lim_{n\to\infty} \frac{x^{2n+2}/(n+1)!}{x^{2n}/n!} = \lim_{n\to\infty} \frac{x^2}{(n+1)} = 0$$

(*) thus ϕ_1 has the radius of convergence ∞.

For ϕ_2, by ratio test

$$\lim_{n\to\infty} \frac{2^{n+1} x^{2n+3}/\{(1)(3)\cdots(2n+3)\}}{2^n x^{2n+1}/\{(1)(3)\cdots(2n+1)\}} = \lim_{n\to\infty} \frac{2x^2}{(2n+3)} = 0$$

(**) thus ϕ_2 has the radius of convergence ∞.

(*) and (**) imply ϕ has the radius of convergence ∞.

Problem 2. (25 points) Solve the following initial value problem using the Laplace transform.

$$(*) \quad \frac{d^2y}{dt^2} - 4y = u_1(t)e^{t-1}; \quad y(0) = 1, \; y'(0) = 0$$

Recall that the unit step function $u_1(t)$ is defined by

$$u_1(t) := \begin{cases} 0 & t < 1 \\ 1 & t \geq 1 \end{cases}.$$

Let $\quad Y(s) := \mathcal{L}\{y(t)\}.$

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s) - 1$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s$$

Take the Laplace transform of both sides of $(*)$

$$\underbrace{s^2Y(s) - s - 4Y(s)}_{(s^2-4)Y(s) - s} = e^{-s}\frac{1}{s-1}$$

$$Y(s) = e^{-s}\underbrace{\frac{1}{(s-1)(s-2)(s+2)}}_{①} + \underbrace{\frac{s}{(s-2)(s+2)}}_{\cancel{s^2-4}\ ②}$$

Partial fractions

① $$\frac{1}{(s-1)(s-2)(s+2)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+2}$$

$$1 = A(s^2-4) + B(s-1)(s+2) + C(s-1)(s-2)$$

$$= s^2(A+B+C) + s(B-3C) + (-4A-2B+2C)$$

$B = 3C$ / $A+B+C=0$, $A = -4C$ / $-4A-2B+2C = 1$, $C = 1/12$

$B = 1/4$ $\qquad A = -1/3$

② $$\frac{s}{(s-2)(s+2)} = \frac{A}{(s-2)} + \frac{B}{(s+2)}$$

$$s = A(s+2) + B(s-2)$$

$$= s(A+B) + (2A-2B)$$

$A+B=1$ / $A=B$

$B = 1/2$ $\qquad A = 1/2$

Thus

$$Y(s) = e^{-s}\left\{\frac{1}{4}\,\frac{1}{s-2} - \frac{1}{3}\,\frac{1}{s-1} + \frac{1}{12}\,\frac{1}{s+2}\right\}$$

$$+ \frac{1}{2}\left\{\frac{1}{s-2} + \frac{1}{s+2}\right\}$$

$$\underline{y}(t) = \frac{1}{4}\,u_1(t)\,e^{2(t-1)} - \frac{1}{3}\,u_1(t)\,e^{t-1} + \frac{1}{12}\,u_1(t)\,e^{-2(t-1)}$$

$$+ \frac{1}{2}\,e^{2t} + \frac{1}{2}\,e^{-2t}$$

Problem 3. (15 points) This question concerns the initial value problem

$$(++)\quad \frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = f(t); \quad y(0) = 0,\ y'(0) = 0$$

where $a, b \in \mathbb{R}$ are constants. Above, $f : \mathbb{R} \to \mathbb{R}$ is a given continuous function such that $0 < f(t) < e^{ct}$ for all $t \in \mathbb{R}$ and for some constant $c \in \mathbb{R}$. Suppose that

$$y(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

is the solution of this initial value problem for $g(t) = e^{2t}\sin(t)$. Determine the values of the constants a, b.

Let $Y(s) := \mathcal{L}\{y(t)\}$ and $F(s) := \mathcal{L}\{f(t)\}$, $G(s) := \mathcal{L}\{g(t)\}$. Furthermore the solution $y(t)$ satisfies

$$y(t) = f(t) * g(t)$$
$$Y(s) = F(s)\,G(s).$$

Take the Laplace transform of both sides of $(++)$ keeping in mind $y(0) = y'(0) = 0$

$$s^2 Y(s) + a\cdot s\cdot Y(s) + b\cdot Y(s) = F(s)$$
$$(s^2 + as + b)\,Y(s) = F(s)$$
$$(s^2 + as + b)\,\cancel{F(s)}\,G(s) = \cancel{F(s)} \quad \left(\begin{array}{l}\text{Note: } F(s) \neq 0 \\ \text{since } f(t) > 0 \\ \forall t\end{array}\right.$$
$$G(s) = \frac{1}{s^2 + as + b} = \mathcal{L}\{e^{2t}\sin t\}$$
$$= \frac{1}{(s-2)^2 + 1} = \frac{1}{s^2 - 4s + 5}$$

Thus $a = -4$ and $b = 5$.

Problem 4.

(a) (15 points) Find the general solution of the following homogeneous system.

$$\frac{dx_1}{dt} = -7x_1 + 6x_2$$
$$\frac{dx_2}{dt} = 6x_1 + 2x_2$$

The homogeneous system can be expressed as

$$\frac{dx}{dt} = \underbrace{\begin{bmatrix} -7 & 6 \\ 6 & 2 \end{bmatrix}}_{A} x$$

where $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

Need to find the eigenvalues of A

$$p(\lambda) = \det(A - \lambda I_2)$$
$$= (-7-\lambda)(2-\lambda) - 36 = \lambda^2 + 5\lambda - 50$$
$$= (\lambda+10)(\lambda-5)$$

Eigenvalues: $\lambda_1 = 5$ $\lambda_2 = -10$

corresponding eigenvectors: $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Thus the general solution is given by

$$c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

$$= c_1 \begin{bmatrix} e^{5t} \\ 2e^{5t} \end{bmatrix} + c_2 \begin{bmatrix} -2e^{-10t} \\ e^{-10t} \end{bmatrix}$$

for arbitrary $c_1, c_2 \in \mathbb{R}$.

(b) (20 points) Find the general solution of the following nonhomogeneous system.

$$\frac{dx_1}{dt} = -7x_1 + 6x_2 + e^{-3t}$$
$$\frac{dx_2}{dt} = 6x_1 + 2x_2 + 2e^{-3t}$$

(Note: Part (a) concerns the associated homogeneous system.)

Method of diagonalization:

(You can also use variation of parameters, or Laplace transform)

From part (a)

$$\underbrace{\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^{-1}}_{V^{-1}} \underbrace{\begin{bmatrix} -7 & 6 \\ 6 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} 5 & 0 \\ 0 & -10 \end{bmatrix}}_{\Lambda}$$

The system can be expressed as

$$\frac{dx}{dt} = \begin{bmatrix} -7 & 6 \\ 6 & 2 \end{bmatrix} x + \begin{bmatrix} e^{-3t} \\ 2e^{-3t} \end{bmatrix}.$$

Let $x = Vy$. We have

$$\frac{dy}{dt} = V^{-1}\begin{bmatrix} -7 & 6 \\ 6 & 2 \end{bmatrix} Vy + V^{-1}\begin{bmatrix} e^{-3t} \\ 2e^{-3t} \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & -10 \end{bmatrix} y + \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} e^{-3t} \\ 2e^{-3t} \end{bmatrix}$$

$$\frac{dy_1}{dt} = 5y_1 + e^{-3t} \qquad \left(\frac{dy_1}{dt} - 5y_1 = e^{-3t} \qquad \frac{d\{y_1 e^{-5t}\}}{dt} = e^{-8t} \right.$$

$$\frac{dy_2}{dt} = -10y_2$$

Thus the solution is given by

$$y_2(t) = c_2 e^{-10t}$$

$$y_1(t) = -\frac{e^{-3t}}{8} + c_1 e^{5t}$$

for any $c_1, c_2 \in \mathbb{R}$.

We look for any particular solution. Thus choose $c_1 = c_2 = 0$, and notice

$$x = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -e^{-3t}/8 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-3t}/8 \\ -2e^{-3t}/8 \end{bmatrix}$$

The general solution is given by

$$c_1 \begin{bmatrix} e^{5t} \\ 2e^{5t} \end{bmatrix} + c_2 \begin{bmatrix} -2e^{-10t} \\ e^{-10t} \end{bmatrix} + \begin{bmatrix} -e^{-3t}/8 \\ -2e^{-3t}/8 \end{bmatrix}$$

for arbitrary $c_1, c_2 \in \mathbb{R}$.