Solution Sheet

MATH 204: Differential Equations

Midterm 1 - Fall 2014

Duration : 110 minutes

#1	15	
#2	20	
#3	15	
#4	15	
#5	35	
Σ	100	

NAME ____________________

STUDENT ID ____________________

SIGNATURE ____________________

- Put your name, student ID and signature in the space provided above.
- No calculators or any other electronic devices are allowed.
- This is a closed-book and closed-notes exam.
- Show all of your work; full credit will not be given for unsupported answers.
- Write your solutions clearly; no credit will be given for unreadable solutions.
- Mark your section below.

SECTION 1 (EMRE MENGI MW 10:00-11:15) ____________________

SECTION 2 (ALI ÜLGER, TUTH 14:30-15:45) ____________________

SECTION 3 (ALI MOSTAFAZADEH, MW 13:00-14:15) ____________________

SECTION 4 (ALTAN ERDOĞAN, TUTH 13:00-14:15) ____________________

Question 1. (15 points) Find the general solution of the following differential equation on the indicated interval.

$$\frac{dy}{dt} = 2\cos(t) + y\tan(t), \quad 0 < t < \pi/2$$

$$\frac{dy}{dt} - y \tan t = 2\cos t$$

General solution is given by

$$\frac{1}{\mu(t)} \left[\int \mu(t)(2\cos t)\, dt + c \right]$$

where

$$\mu(t) = e^{-\int \tan t \, dt}$$

$$= e^{\ln|\cos t| + d}$$

(choose $d=0$ you are free to choose d whatever you like)

$$\underline{\mu(t) = e^{\ln|\cos t|}} = |\cos t| = \cos t$$

$\downarrow$

$t \in (0, \pi/2)$

General solution

$$\frac{1}{\cos t} \left[\int \underbrace{2\cos^2 t}_{= \cos(2t)+1} \, dt + c \right] =$$

$$\frac{1}{\cos t} \left[\frac{\overset{=2\sin t \cos t}{\sin(2t)}}{2} + t + c \right] =$$

$$\sin t + t/\cos t + c/\cos t$$

for arbitrary $c \in \mathbb{R}$.

Question 2. (20 points) This question concerns the following differential equation.

$$\underbrace{\{x + \sin(y)\}}_{M(x,y)} + \underbrace{\{x\cos(y) - 2y\}}_{N(x,y)}\frac{dy}{dx} = 0$$

Show that this equation is exact and find its solution.

$M(x,y) = x + \sin(y) \qquad N(x,y) = x\cos y - 2y$

$$\frac{\partial M(x,y)}{\partial y} = \cos y \qquad \frac{\partial N(x,y)}{\partial x} = \cos y$$

since $\dfrac{\partial M(x,y)}{\partial y} = \dfrac{\partial N(x,y)}{\partial x}$,

the equation is exact.

The solution should be given by

$$\psi(x,y) = \int M(x,y)\,dx + h(y)$$

$$= \frac{x^2}{2} + x \cdot \sin y + h(y)$$

where $h(y)$ is such that

$$\frac{\partial \psi(x,y)}{\partial y} = N(x,y)$$

$$x \cdot \cos y + h'(y) = x \cdot \cos y - 2y$$

$$\Longrightarrow h'(y) = -2y \quad \Longrightarrow h(y) = -y^2 + d$$

free to choose for instance $d=0$

Thus any solution $y = y(x)$ satisfies

$\psi(x,y) = c$, that is $\dfrac{x^2}{2} + x \cdot \sin y - y^2 = c$

for some $c \in \mathbb{R}$.

Question 3. Suppose that $p(t)$ is such that the Wronskian of any two solutions of the equation

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} - \frac{1}{t}y = 0, \quad t > 0 \tag{1}$$

is also a solution. Furthermore, suppose that $p(1) = 0$.

(a) (10 points) Find $p(t)$.

By Abel's thm. $W(y_1, y_2)(t) = C \cdot e^{-\int p(t)dt}$
for any solutions y_1, y_2. Notice

$$[W(y_1,y_2)(t)]' = -C \cdot e^{-\int p(t)dt} \cdot p(t)$$

$$[W(y_1,y_2)(t)]'' = C \cdot e^{-\int p(t)dt} \cdot [p(t)]^2 - C \cdot e^{-\int p(t)dt} \cdot p'(t)$$

Since Wronskian is a solution for $t>0$

$$[W(y_1,y_2)(t)]'' + p(t)[W(y_1,y_2)(t)]' - \frac{1}{t}W(y_1,y_2)(t) = 0$$

$$\Longrightarrow \{C\cdot[p(t)]^2 - C\cdot p'(t)\} + p(t)\{-C p(t)\} - \frac{C}{t} = 0 \quad t>0$$

$$\Longrightarrow p'(t) = -\frac{1}{t}$$

Thus $p(t) = -\ln t + c$
Using $p(1)=0$, we have $c=0$
that is $p(t) = -\ln t$

(b) (5 points) Write down a nonzero solution of (1).

Since Wronskian is a solution, by Abel's thm

$$C e^{-\int p(t)dt} = C \cdot e^{\int \ln t \, dt} \overset{\text{See } (*)}{=} C \cdot e^{\ln t \cdot t - t}$$

for instance

$$\phi(t) = e^{\ln t \cdot t - t}$$

is a nonzero solution

[redacted] (*)

$$\int \ln t \, dt \overset{\text{integration by parts}}{=} \ln t \cdot t - \int 1 \cdot dt = \ln t \cdot t - t$$

$u = \ln t$ $dv = dt$
$du = \frac{1}{t}dt$ $v = t$

Question 4.

(a) (5 points) Write down the general solution of the following differential equation.

$$25\frac{d^2y}{dt^2} - 10\frac{dy}{dt} + y = 0$$

$p(r) = 25r^2 - 10r + 1 = (5r-1)^2$
has a repeated root $r_1 = r_2 = 1/5$

Thus the general solution is given by

$$c_1 \cdot e^{t/5} + c_2 \cdot t\, e^{t/5}$$

for arbitrary $c_1, c_2 \in \mathbb{R}$.

(b) (10 points) Write down an initial value problem of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0, \quad y(0) = 2, \;\; y'(0) = -1$$

for some constants $a, b, c \in \mathbb{R}$ such that its solution satisfies

$$y(t) \to 1 \quad \text{as} \quad t \to \infty.$$

In particular, you need to specify the values of the constants $a, b, c \in \mathbb{R}$.

Solution must be of the form

$$y(t) = c_1 e^{-\alpha t} + 1 \quad \left(y'(t) = -c_1 \alpha e^{-\alpha t}\right)$$

for some real scalar $\alpha > 0$ and $c_1 \in \mathbb{R}$, as otherwise $y(t) \to 1$ as $t \to \infty$ is not possible.

Imposing initial conditions

$$y(0) = 2 \implies c_1 + 1 = 2 \implies c_1 = 1$$

$$y'(0) = -1 \implies -\alpha c_1 = -1 \implies \alpha = 1$$

Thus

$$y(t) = e^{-t} + 1.$$

The polynomial $p(r) = ar^2 + br + c$ must have the roots $r_1 = 0$, $r_2 = -1$.
For instance $p(r) = t(t+1) = t^2 + t$, that is
$a = b = 1 \quad c = 0$

Question 5.

(a) (15 points) The function $y_1(t) = t^2$ is a solution of the homogeneous equation

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} - 4y = 0, \quad t > 0.$$

Find another solution $y_2(t)$ such that $y_1(t), y_2(t)$ form a fundamental set of solutions for this homogeneous equation.

Use the method of reduction of order.

Suppose $y_2(t) = v(t)t^2$ is another solution.

$$y_2'(t) = v'(t)\,t^2 + v(t)\,2t$$

$$y_2''(t) = v''(t)\,t^2 + v'(t)\,2t + v'(t)\cdot 2t + 2v(t)$$

$$= t^2 v''(t) + 4t\,v'(t) + 2v(t)$$

Plug these in the ODE

$$t^2\{t^2 v''(t) + 4t\,v'(t) + \cancel{2v(t)}\} +$$

$$t\,\{t^2 v'(t) + \cancel{2t\,v(t)}\} - \cancel{4t^2 v(t)} = 0$$

$$t^4 v''(t) = -5t^3 v'(t)$$

Letting $w(t) = v'(t)$, we have

$$\frac{1}{w(t)}\,w'(t) = -\frac{5}{t}$$

$$\ln|w(t)| = -5\ln|t| = \ln(1/t^5)$$

$$w(t) = 1/t^5$$

Thus $v(t) = -1/(4t^4) + c$ for any constant c.

Choose $c = 0$ so that

$$y_2(t) = \cancel{\left(-\frac{1}{4}\right)}\cdot\frac{1}{t^2}$$

(Note: we can drop $-1/4$ here, since if ϕ is a solution, so is $c\cdot\phi$ for constant c)

is another solution. Finally,

$$W(y_1, y_2)(t) = \det\left(\begin{bmatrix} t^2 & 1/t^2 \\ 2t & -2/t^3 \end{bmatrix}\right) = -\frac{2}{t} - \frac{2}{t} = -\frac{4}{t} \neq 0$$

Thus y_1, y_2 form a fundamental set of solutions.

(b) (20 points) Find the general solution (the set of all solutions) of the following differential equation.

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} - 4y = t^2 - 1,\ \ t > 0$$

Rewrite the differential eqn. of the form

$$\frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt} - \frac{4}{t^2}y = \underbrace{\boxed{\frac{t^2-1}{t^2}}}_{r(t) :=}$$

We use the method of variation of parameters.

A particular solution is given by

$$\phi(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$$

where $y_1(t) = t^2$ and $y_2(t) = 1/t^2$ are solutions for the homogeneous equation with Wronskian

$$W(y_1, y_2)(t) = -4/t.$$

Furthermore

$$v_1(t) = -\int \frac{r(t)\,y_2(t)}{W(y_1,y_2)(t)}\,dt = -\int\left(\frac{t^2-1}{t^2}\right)\cdot\frac{(1/t^2)}{(-4/t)}\,dt$$

$$= \frac{1}{4}\int \frac{t^2-1}{t^3}\,dt \underset{\substack{\downarrow \\ \text{set integration} \\ \text{constant } 0}}{=} \frac{1}{4}\left\{\ln|t| + \frac{1}{2t^2}\right\}$$

and

$$v_2(t) = \int \frac{r(t)\,y_1(t)}{W(y_1,y_2)(t)}\,dt = \int\left(\frac{t^2-1}{t^2}\right)\cdot\frac{t^2}{(-4/t)}\,dt$$

$$= -\frac{1}{4}\int t^3 - t\,dt \underset{\substack{\downarrow \\ \text{set integration} \\ \text{constant } 0}}{=} -\frac{1}{4}\left\{\frac{t^4}{4} - \frac{t^2}{2}\right\}$$

General solution is given by

$$\phi(t) + c_1 y_1(t) + c_2 y_2(t)$$

$$= \frac{1}{4}\left\{\ln|t| + \frac{1}{2t^2}\right\}t^2 - \frac{1}{4}\left\{\frac{t^4}{4} - \frac{t^2}{2}\right\}\frac{1}{t^2} + c_1 t^2 + \frac{c_2}{t^2}$$

for arbitrary $c_1, c_2 \in \mathbb{R}$.