Math 107: Midterm Exam # 2
May 5, 2018

Problem 1 Let $\mathbf{A} := \begin{bmatrix} 1 & 1 & 0 & -3 \\ 1 & -1 & 2 & 0 \\ 2 & 1 & 0 & 1 \\ 2 & 0 & 0 & -3 \end{bmatrix}$.

1.a (10 points) Compute the determinant of $\mathbf{A}$ and find out if it is an invertible matrix.

1.b (5 points) What is the rank of $\mathbf{A}$? Why?

Problem 2 (15 points)Use Cramer's rule to solve the following system of equations.

$$\begin{aligned} x - 2y - z &= -1 \\ 4x + y - 2z &= 1 \\ 2x + 4y - z &= 2 \end{aligned}$$

Warning: Solving the system without using Cramer's rule will not earn you any credit.

Problem 3 Let V be a real vector space and o be the zero vector in V. Prove the following statements.

3.a (5 points) For all $v \in V$, $0.v = o$.

3.b (5 points) For all $\alpha \in \mathbb{R}$, $\alpha.o = o$.

Problem 4 State the definition of the following terms.

4.a (4 points) Span of a nonempty subset A of a vector space:

4.b (3 points) A finite-dimensional real vector space:

4.c (3 points) A basis of a vector space:

4.d (5 points) A linear transformation $T : V \to W$ where V and W are vector spaces.

Problem 5 Let V be the vector space of 2×2 matrices. Give an example of the following objects:

5.a (10 points) Three elements $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ of V such that $\{\mathbf{A}, \mathbf{B}\}$ and $\{\mathbf{B}, \mathbf{C}\}$ are linearly independent, but $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ is linearly-dependent. Justify your response.

5.b (10 points) A linear transformation $T : V \to V$ whose null space is two-dimensional. You do not need to show that T is linear, but must find its null space and explain why it is two-dimensional.

5.c (5 points) An onto linear transformation $T : V \to \mathbb{R}^2$ satisfying:

$$T(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad T(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

5.d (10 points) Prove that your response to Problem 5.c is actually a linear transformation that is onto.

Problem 6 (10 points) Let V be a vector space, U and W be subspaces of V, and $U \cap W$ denote the intersection of U and W, i.e., $U \cap W := \{v \in V \mid v \in U \text{ and } v \in W\}$. Prove that $U \cap W$ is a subspace of V.

1) a) $A = \begin{pmatrix} 1 & 1 & 0 & -3 \\ 1 & -1 & 2 & 0 \\ 2 & 1 & 0 & 1 \\ 2 & 0 & 0 & -3 \end{pmatrix}$

$$\det A = -2\big(2 - 3\cdot(-2)\big) - 3\cdot\big(-2 - 1\cdot(-4)\big)$$
$$= -16 - 6 = -22.$$

So A is invertible.

b) Rank $A = 4$ since A is invertible so all columns are lin. independent.

2)
$$x - 2y - z = -1$$
$$4x + y - 2z = 1$$
$$2x + 4y - z = 2$$

$A = \begin{pmatrix} 1 & -2 & -1 \\ 4 & 1 & -2 \\ 2 & 4 & -1 \end{pmatrix}$

$$\det A = 7 + 2\cdot 0 - 1\cdot 14 = -7$$

$$\det \begin{pmatrix} -1 & -2 & -1 \\ 1 & 1 & -2 \\ 2 & 4 & -1 \end{pmatrix} = -1\cdot 7 + 2\cdot 3 - 1\cdot 2 = -3$$

$$\det \begin{pmatrix} 1 & -1 & -1 \\ 4 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix} = -1 + 4 - 6 = -3$$

$$\det \begin{pmatrix} 1 & -2 & -1 \\ 4 & 1 & 1 \\ 2 & 4 & 2 \end{pmatrix} = -2 + 2\cdot 6 - 14 = -4$$

$x = \frac{3}{7}$ $\quad y = \frac{3}{7}$ $\quad z = \frac{4}{7}$

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3) a) Show $0\cdot v = 0$

$$0\cdot v = (0+0)\cdot v = 0\cdot v + 0\cdot v \Rightarrow 0\cdot v = 0v - 0\cdot v = 0$$

b) Show $\alpha\cdot 0 = 0$.

$$\alpha\cdot 0 = \alpha(0+0) = \alpha\cdot 0 + \alpha\cdot 0 \Rightarrow$$

$$\alpha \cdot 0 = \alpha \cdot 0 - \alpha \cdot 0 = 0$$

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4) a) Span of a nonempty subset A of a v.s. is $= \{ c_1 v_1 + \dots + c_k v_k : v_1, \dots, v_k \in A, c_1, \dots, c_k \in \mathbb{R} \}$

b) A finite dimensional real vector space is a vector space V with a finite basis $B = \{v_1, \dots, v_k\}$

c) A basis B of a vector space V is a linearly independent set such that $Span(B) = V$.

d) $T: V \to W$ is called a linear trans. if

$T(v_1 + v_2) = T(v_1) + T(v_2)$ for each $v_1, v_2 \in V$

$T(\alpha \cdot v) = \alpha \cdot T(v)$ for each $v \in V$, $\alpha \in \mathbb{R}$.

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5) a) $V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$

Let $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

Then if $c_1 \cdot A + c_2 \cdot B = \begin{pmatrix} 0 & 0 \\ c_1 & 0 \end{pmatrix} + \begin{pmatrix} c_2 & c_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow c_1 = c_2 = 0$ So $\{A, B\}$ are independent.

If $c_1 \cdot B + c_2 \cdot C = \begin{pmatrix} c_1 & c_1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} c_2 & c_2 \\ c_2 & 0 \end{pmatrix}$

$= \begin{pmatrix} c_1 + c_2 & c_1 + c_2 \\ c_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow c_2 = 0 \Rightarrow c_1 = 0$ So $\{B, C\}$ are independent.

But $A + B = C$ So $\{A, B, C\}$ are dependent.

b) Let $T : V \longrightarrow V$

$\begin{pmatrix} a & b \\ \dots \end{pmatrix} \longmapsto \begin{pmatrix} 0 & b \\ \dots \end{pmatrix}$

$$\begin{pmatrix} c & d \end{pmatrix} \quad \begin{pmatrix} 0 & d \end{pmatrix}$$

$$Null(T) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : b = d = 0 \right\}$$

$$= Span\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ are independent.}$$

So $Null(T)$ is two dimensional.

c) $T: V \to \mathbb{R}^2$ $\quad T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad T\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

and want T to be onto.

Let $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \\ \frac{a+d}{2} \end{pmatrix}$.

i) T is linear since $T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right)$

$$= \begin{pmatrix} a+e \\ \frac{a+e+d+h}{2} \end{pmatrix} = T\begin{pmatrix} a & b \\ c & d \end{pmatrix} + T\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$\begin{pmatrix} a+e \\ \frac{a+e}{2} + \frac{d+h}{2} \end{pmatrix}$$

& $T\left(k\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} k \cdot a \\ \frac{ka + k \cdot d}{2} \end{pmatrix} = k\begin{pmatrix} a \\ \frac{a+d}{2} \end{pmatrix} = k \cdot T\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

T is onto since for any $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{R}^2$,

$$T\begin{pmatrix} \alpha & 0 \\ 0 & 2\beta - \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \frac{\alpha + 2\beta - \alpha}{2} \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

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6) $0 \in U$, $0 \in W \Rightarrow 0 \in U \cap W$

If $v_1, v_2 \in U \cap W \Rightarrow v_1, v_2 \in U \Rightarrow v_1 + v_2 \in U$

$v_1, v_2 \in W \Rightarrow v_1 + v_2 \in W$

$\Rightarrow v_1 + v_2 \in U \cap W$.

If $v \in U \cap W$, $\alpha \in \mathbb{R}$, $v \in U \Rightarrow \alpha \cdot v \in U$

$v \in W \Rightarrow \alpha \cdot v \in W$

$\Rightarrow \alpha \cdot v \in U \cap W$.