Solutions to Midterm Exam 1

Problem 1 Consider the system of equations:

$$x_1 - x_2 + x_3 = 0,$$

$$2x_1 + x_2 + x_3 = 0,$$

$$x_1 - 2x_2 + s\,x_3 = 0,$$

$$3x_1 + s\,x_2 + t\,x_3 = 0,$$

where x_1, x_2, and x_3 are unknowns and r and s numbers.

(15 points) Find all values of s and t such that this system has a nontrivial solution.

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & -2 & s & 0 \\ 3 & s & t & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & -1 & s-1 & 0 \\ 0 & s+3 & t-3 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & s-\frac{4}{3} & 0 \\ 0 & 0 & \frac{s+3}{3}+t-3 & 0 \end{bmatrix}$$

We have a nontrivial solution iff

$$s - \frac{4}{3} = 0 \qquad \text{and} \qquad \frac{s+3}{3} + t - 3 = 0$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\boxed{s = \frac{4}{3}} \qquad\qquad t = 3 - \frac{s}{3} - 1 = 2 - \frac{s}{3} = 2 - \frac{4}{9} \Rightarrow$$

$$\boxed{t = \frac{14}{9}}$$

Problem 2 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

a (10 points) Prove that T is one-to-one if and only if $\mathrm{Nul}(T)$ is trivial, i.e., $\mathrm{Nul}(T) = \{\mathbf{0}\}$.

$\Rightarrow$ If T is 1-to-1, $\forall\, x \in Nul(T)$, $Tx = 0 = T0$

$\Rightarrow$ $x = 0$ because T is 1-to-1 $\Rightarrow$ $Nul(T) \subseteq \{0\}$

But $0 \in Nul(T) \Rightarrow \{0\} \subseteq Nul(T)$ $\Rightarrow$ $Nul(T) = \{0\}$

$\Leftarrow$ If $Nul(T) = \{0\}$, $\forall\, x_1, x_2 \in \mathbb{R}^n$

$T(x_1) = T(x_2) \Rightarrow T(x_1) - T(x_2) = 0 \Rightarrow$

$T(x_1 - x_2) = 0 \Rightarrow x_1 - x_2 \in Nul(T)$

but $Nul(T) = \{0\}$ $\Rightarrow$ $x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2 \Rightarrow T$ is 1-to-1. $\boxtimes$

b (10 points) Show that the range (image) of T is a subspace of $\mathbb{R}^m$.

1) $T(0) = 0 \Rightarrow 0 \in Ran(T) \subseteq \mathbb{R}^m$

2) let $y_1, y_2 \in Ran(T) \Rightarrow \exists\, x_1, x_2 \in \mathbb{R}^n$,

$T(x_1) = y_1$, $T(x_2) = y_2 \Rightarrow y_1 + y_2 = T(x_1) + T(x_2)$

$= T(x_1 + x_2)$

$\Rightarrow y_1 + y_2 \in Ran(T)$

3) let $\alpha \in \mathbb{R}$, $y \in Ran(T) \Rightarrow \exists\, x \in \mathbb{R}^n$, $T(x) = y$

$\Rightarrow \alpha y = \alpha T(x) = T(\alpha x) \Rightarrow \alpha y \in Ran(T)$

①, ②, ③ $\Rightarrow$ $Ran(T)$ is a subspace of $\mathbb{R}^m$.

Problem 3 Consider the row operation that replaces the second row of every $3 \times n$ matrix by the sum of this row and twice the first row.

a (10 points) For $n = 1$ this defines a function $E : \mathbb{R}^3 \to \mathbb{R}^3$. Give the formula for $E(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix})$, and show that E is a linear transformation.

$E\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2+2x_1 \\ x_3 \end{bmatrix}$. Let $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{R}^3 \Rightarrow$

$E(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}) = E(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}) = \begin{bmatrix} x_1+y_1 \\ x_2+y_2+2(x_1+y_1) \\ x_3+y_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2+2x_1 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2+2y_1 \\ y_3 \end{bmatrix}$

$= E(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) + E(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix})$ ①

Let $\alpha \in \mathbb{R}$, $E(\alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = E(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix}) = \begin{bmatrix} \alpha x_1 \\ \alpha x_2+2\alpha x_1 \\ \alpha x_3 \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ x_2+2x_1 \\ x_3 \end{bmatrix} = \alpha E\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ ②

① & ② $\Rightarrow$ E is linear.

b (5 points) Determine the standard matrix for E.

$E\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. $E\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $E\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow \mathbb{E} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c (5 points) Find the inverse of E, i.e., give the formula for $E^{-1}(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix})$.

$[\mathbb{E} \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

$\Rightarrow \mathbb{E}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E^{-1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_1+x_2 \\ x_3 \end{bmatrix}$

Problem 4 Consider the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & a \\ 2 & -4 & -1 \\ 3 & 0 & 2a \end{bmatrix}.$$

a (10 points) Use elementary row operations to find all values of a for which $\mathbf{A}$ is invertible.

$$\left[\begin{array}{ccc|ccc} -1 & 2 & a & 1 & 0 & 0 \\ 2 & -4 & -1 & 0 & 1 & 0 \\ 3 & 0 & 2a & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} -1 & 2 & a & 1 & 0 & 0 \\ 0 & 0 & 2a-1 & 2 & 1 & 0 \\ 0 & 6 & 5a & 3 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} -1 & 2 & a & 1 & 0 & 0 \\ 0 & 6 & 5a & 3 & 0 & 1 \\ 0 & 0 & 2a-1 & 2 & 1 & 0 \end{array}\right]$$

To be invertible $2a-1 \neq 0 \Rightarrow \boxed{a \neq \frac{1}{2}}$

b (10 points) Use elementary row operations to determine $\mathbf{A}^{-1}$ for $a = 0$.

For $a = 0$

$$= \left[\begin{array}{ccc|ccc} -1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 6 & 0 & 3 & 0 & 1 \\ 0 & 0 & -1 & 2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{6} \\ 0 & 0 & 1 & -2 & -1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{6} \\ 0 & 0 & 1 & -2 & -1 & 0 \end{array}\right]$$

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{6} \\ -2 & -1 & 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 0 & 0 & 2 \\ 3 & 0 & 1 \\ -12 & -6 & 0 \end{bmatrix}$$

Problem 5 Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}\right) := \begin{bmatrix} 2x + y \\ z \\ x + z + t \end{bmatrix}.$$

a (5 points) Find a basis for Nul(T).

$$\underbrace{T\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}}_{=\,\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} 2x + y = 0 \\ \boxed{z = 0} \\ x + z + t = 0 \end{array} \Rightarrow \begin{array}{l} \boxed{y = -2x} \\ \boxed{t = -x} \end{array}$$

$$\Rightarrow \mathbf{x} = \begin{bmatrix} x \\ -2x \\ 0 \\ -x \end{bmatrix} = x\begin{bmatrix} 1 \\ -2 \\ 0 \\ -1 \end{bmatrix}$$

so $\left\{\begin{bmatrix} 1 \\ -2 \\ 0 \\ -1 \end{bmatrix}\right\}$ is a basis of Nul(T).

b (5 points) Find the standard matrix form of T and denote it by **A**.

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

c (10 points) Let $S : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation defined by $S(\mathbf{x}) := \mathbf{A}^T\mathbf{x}$, where $\mathbf{A}^T$ is the transpose of $\mathbf{A}$. Find a basis for the range (image) of S.

$$S\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{A^T} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_3 \\ x_2 \\ x_2 + x_3 \\ x_3 \end{bmatrix}$$

$$\text{Ran}(S) = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$S\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} 2x_1 + x_3 = 0 \Rightarrow \boxed{x_1 = 0} \\ \boxed{x_2 = 0} \\ x_2 + x_3 = 0 \\ \boxed{x_3 = 0} \end{array}$$

So $S(\mathbf{x}) = \mathbf{0}$ has a unique sol. $\Rightarrow$ columns of A^T are linearly indep. $\Rightarrow \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly-indep.

Hence it is basis.

d (5 points) Determine whether S is one-to-one, and explain why.

We showed that $S(\mathbf{x}) = \mathbf{0}$ has no nontrivial solution. This implies that S is 1-to-1.