Math 107, Fall 2012, Quiz # 5a

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (8 points)Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator given by $L(x, y) := (x - y, \cancel{0}, x + y)$. Determine the matrix representation of L in the bases (E, F), where $E := \{(2, 0), (0, 3)\}$ and $F := \{(1, 2), (2, 1)\}$ are the different bases of $\mathbb{R}^2$.

$$L(2,0) = (2-0, 2+0) = (2,2) = \frac{2}{3}(1,2) + \frac{2}{3}(2,1)$$

$$L(0,3) = (0-3, 0+3) = (-3,3) = 3(1,2) - 3(2,1)$$

$$\Rightarrow \quad M = \begin{bmatrix} 2/3 & 3 \\ 2/3 & -3 \end{bmatrix}$$

Problem 2 (12 points) Let $L : \mathbb{C}^2 \to \mathbb{C}^2$ be the every-where defined linear operators defined by $L(w, z) := ((4 + 2i)w - 2iz, (1 + 3i)w + (1 - i)z)$.

2.a) Find a basis for $Nul(L)$.

Quiz 5 / 2f

2.b) Find a basis for $Ran(L)$.

Quiz 5 / 2f

2.c) Find $\alpha \in \mathbb{C}$ such that the equation $L\vec{x} = (1, \alpha)$ has a solution. For the value you find, determine if $L\vec{x} = (1, \alpha)$ has a unique solution.

$(1,\alpha) \in Ran(L) \iff L\vec{x} = (1,\alpha)$ has a solution.

$\{(-2i, 1-i)\}$ is a basis of $Ran(L)$.

$(1,\alpha) \in Ran(L) \iff \exists \beta \in \mathbb{C}$ s.t $\beta(-2i, 1-i) = (1,\alpha)$

$\iff -2\beta i = 1 \; (\beta = -\frac{1}{2i})$
$(1-i)\beta = \alpha$

$\iff \alpha = \frac{1+i}{2}$

$\Rightarrow$ for $\alpha = \frac{1+i}{2}$, $L\vec{x} = (1,\alpha)$ has a solution.

Since $Nul(L)$ is not trivial, $L\vec{x} = (1,\alpha)$ has more solutions.

Math 107, Fall 2012, Quiz # 5b

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (5 points) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the every-where defined linear operator defined by $L(x, y) := (x - y, -x + y)$. Adress the existence and uniqueness problems for the linear equation $L\vec{x} = (2, 0)$.

$(2,0) \in \text{Ran}(L) \Leftrightarrow L\vec{x} = (2,0)$ has one or more solutions.

$(2,0) \notin \text{Ran}(L)$. If $(2,0) \in \text{Ran}(L)$, there would be $(x,y) \in \mathbb{R}^2$ s.t $L(x,y) = (2,0)$, that

is $x - y = 2$, contradiction.

$-x+y=0$, i.e, $x-y=0$

So $L\vec{x} = (2,0)$ has no solution

Problem 2 (15 points)Let $D : \mathcal{P}_3(\mathbb{R},\mathbb{R}) \to \mathcal{P}_3(\mathbb{R},\mathbb{R})$ be the derivative operator and $\mathcal{B} = \{p_0, p_1, p_2, p_3\}$ and $\tilde{\mathcal{B}} := \{q_0, q_1, q_2, q_3\}$ be different bases of $\mathcal{P}_3(\mathbb{R},\mathbb{R})$ such that $p_0(x) := 1,\ p_1(x) := x,\ p_2(x) := x^2, p_3(x) := x^3$ for $x \in \mathbb{R}$ and $q_0 := p_0,\ q_1 := p_1 + p_2,\ q_2 := p_1 - p_2,\ q_3 := p_3$.

2.a) Find the matrix representation of D in the bases $(\mathcal{B}, \tilde{\mathcal{B}})$.

$$\left.\begin{array}{l} D(p_0) = 0 = 0\cdot q_0 + 0\cdot q_1 + 0\cdot q_2 + 0\cdot q_3 \\ D(p_1) = 1 = 1\cdot q_0 + 0\cdot q_1 + 0 q_2 + 0\cdot q_3 \\ D(p_2) = 2x = 2p_1 = 0\cdot q_0 + 1\cdot q_1 + 1\cdot q_2 + 0\cdot q_3 \\ D(p_3) = 3x^2 = 3p_2 = 0\cdot q_0 + \frac{3}{2}q_1 - \frac{3}{2}q_2 + 0\cdot q_3 \end{array}\right\} \Rightarrow D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2.b) Use the matrix representation you find in part (a) to evaluate the derivative of $q : \mathbb{R} \to \mathbb{R}$ defined by $q(x) := x^3 + 2x^2 - 3x + 5$.

the vector representation of $q(x) = x^3 + 2x^2 - 3x + 5$ in $\mathcal{B}$ is $\vec{a} = \begin{bmatrix} 5 \\ -3 \\ 2 \\ 1 \end{bmatrix}$.

$$\Rightarrow \vec{b} = D\vec{a} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 1 & -3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 7/2 \\ 1/2 \\ 0 \end{bmatrix}.$$

$\vec{b}$ is the vector representation of $D(q)$ in $\tilde{\mathcal{B}}$.

$$\Rightarrow q'(x) = -3q_0 + \frac{7}{2}q_1 - \frac{1}{2}q_2 = -3p_0 + \frac{7}{2}(p_1+p_2) - \frac{1}{2}(p_1 - p_2) = -3p_0 + 4p_1 + 3p_2 = -3 + 4x + 3x^2$$

2.c) Use the matrix representation you find in part (a) to evaluate the second derivative of q.

$$q'(x) = 3x^2 + 4x - 3$$

$\Rightarrow$ the vector representation of q' in $\mathcal{B}$ is $a = \begin{bmatrix} -3 \\ 4 \\ 3 \\ 0 \end{bmatrix}$

$$\vec{b} = D\vec{a} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 3 \\ 0 \end{bmatrix}$$

$$\Rightarrow q''(x) = 4q_0 + 3q_1 + 3q_2 = 4p_0 + 3(p_1 + p_2) + 3(p_1 - p_2) = 4p_0 + 6p_1 = 4 + 6x.$$

Math 107, Fall 2012, Quiz # 5c

You have 30 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (8 points) Let $L : \mathbb{C}^2 \to \mathbb{C}^2$ be a linear operator given by $L(w, z) := (w + iz, z - iw)$. Determine the matrix representation of L in the bases (E, F), where $E := \{(1, i), (1, -i)\}$ and $F := \{(1, i), (i, 0)\}$ are different bases of $\mathbb{C}^2$.

$$L(1,i) = (1+i^2, i-i) = (0,0) = 0\cdot(1,i) + 0\cdot(i,0)$$

$$L(1,-i) = (1-i^2, -i-i) = (2,-2i) = -2\cdot(1,i) - 4i\,(i,0)$$

$$\Rightarrow \quad M = \begin{bmatrix} 0 & -2 \\ 0 & -4i \end{bmatrix}$$

Problem 2 (12 points) Let $L : \mathbb{C}^2 \to \mathbb{C}^2$ be the every-where defined linear operators defined by $L(w, z) := (w - iz, w + iz)$.

2.a) Find a basis for $Nul(L)$.

$Nul(L) = \{(w,z) : L(w,z) = (0,0)\}$

So the solutions of the system gives $Nul(L)$:

$$\left.\begin{array}{l} w - iz = 0 \\ w + iz = 0 \end{array}\right\} \Rightarrow 2iz = 0 \Rightarrow \boxed{\begin{array}{l} z = 0 \\ w = 0 \end{array}}$$

So $Nul(L) = \{(0,0)\}$

2.b) Find a basis for $Ran(L)$.

Since $Nul(L) = \{(0,0)\}$, L is 1-1 and as in a finite dimensional vector space, an 1-1 operator is onto, L is onto. So $Ran(L) = \mathbb{C}^2$.

Hence any basis of $\mathbb{C}^2$ is a basis of $Ran(L)$, for example $\{(1,0), (0,1)\}$.

2.c) Find $\alpha \in \mathbb{C}$ such that the equation $L\vec{x} = (1, \alpha)$ has a solution. For the value you find, determine if $L\vec{x} = (1, \alpha)$ has a unique solution.

$(1,\alpha) \in Ran(L) \iff L\vec{x} = (1,\alpha)$ has one or more solution,

Since $Ran(L) = \mathbb{C}^2$, for any value $\alpha \in \mathbb{C}$, $(1,\alpha) \in Ran(L)$.

So $L\vec{x} = (1,\alpha)$ has a solution for all $\alpha \in \mathbb{C}$.

As $Nul(L) = \{(0,0)\}$, $L\vec{x} = (1,\alpha)$ has a unique solution for all $\alpha \in \mathbb{C}$.

Math 107, Fall 2012, Quiz # 5d

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (10 points)Let $L : \mathbb{C}^2 \to \mathbb{C}^2$ be a linear operator given by $L(w, z) := (w + iz, z - iw)$.

1.a) Determine the matrix representation of L in the bases (E, F), where $E := \{(1, i), (1, -i)\}$ and $F := \{(1, i), (i, 0)\}$ are different bases of $\mathbb{C}^2$.

Quiz 5c / Problem 1.

1.b) Let $\vec{x} := (2, 0)$. Use matrix representation of L you find in part(a), to evaluate $L\vec{x}$. *Hint: First write the vector representation of (2,0) in basis E.*

$(2,0) = 1.(1,i) + 1.(1,-i)$, $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$M\vec{a} = \begin{bmatrix} 0 & -2 \\ 0 & 4i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -4i \end{bmatrix} = \vec{b}$

$\vec{b}$ is the vector representation of $L(2,0)$ in basis F.

$\Rightarrow L(2,0) = -2.(1,i) - 4i(i,0) = (2,-2i)$

Problem 2 (10 points) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the every-where defined linear operator defined by $L(x, y) := (x - 2y, 2x + y)$. Adress the existence and uniqueness problems for the following linear equations:

2.a) $L\vec{x} = (2, -2)$

$(2,-2) \in Ran(L) \Leftrightarrow L\vec{x} = (2,-2)$ has a solution.

$(2,-2) \in Ran(L) \Leftrightarrow \exists (x,y) \in \mathbb{R}^2$ s.t. $L(x,y) = (2,-2)$

$\Leftrightarrow \begin{array}{l} x - 2y = 2 \\ 2x + y = -2 \end{array} \Leftrightarrow x = -\frac{2}{5},\ y = \frac{-12}{5}$

$\Rightarrow (2,-2) \in Ran(L) \Rightarrow L\vec{x} = (2,-2)$ has a solution.

Since $Nul(L) = \{(0,0)\}$ (from part b), $L\vec{x} = (2,-2)$ has a unique solution.

2.b) $L\vec{x} = (0, 0)$

$(x,y) \in Nul(L) \Leftrightarrow L(x,y) = (0,0)$

$\Leftrightarrow \begin{array}{l} x - 2y = 0 \\ 2x + y = 0 \end{array}$

$\Leftrightarrow x = y = 0$

So $Nul(L) = \{(0,0)\}$.

Since every element of $Nul(L)$ is a solution of $L\vec{x} = (0,0)$, $L\vec{x} = (0,0)$ has only trivial solution $(0,0)$

Math 107, Fall 2012, Quiz # 5e

You have 30 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (4 points) Let $L : \mathbb{R}^2 \to \mathbb{R}$ be a linear operator given by $L(x, y) := x - 3y$. Determine the matrix representation of L in the bases (E, F), where $E := \{(1, 0), (0, 1)\}$ and $F := \{-3\}$ are the bases of $\mathbb{R}^2$ and $\mathbb{R}$, respectively.

$L(1,0) = 1 - 3 \cdot 0 = 1 = -1/3 \cdot (-3)$

$L(0,1) = 0 - 3 \cdot 1 = -3 = 1 \cdot (-3)$

$\Rightarrow M = \begin{bmatrix} -1/3 & 1 \end{bmatrix}$

Problem 2 (16 points)Let $L : \mathcal{P}_3(\mathbb{R},\mathbb{R}) \to \mathcal{P}_3(\mathbb{R},\mathbb{R})$ be the linear operator defined by $(Lp)(x) := 3\int_0^x p(t)dt - xp(x)$, where $p \in \mathcal{P}_3(\mathbb{R},\mathbb{R})$ and x is an arbitrary real number.

2.a) Find a basis for $Dom(L)$.

Let $p(x) = ax^3 + bx^2 + cx + d \in Dom(L)$.

$p \in Dom(L) \Leftrightarrow Lp \in \mathbb{P}_3(\mathbb{R},\mathbb{R})$.

Then $(Lp)(x) = 3\int_0^x (at^3 + bt^2 + ct + d)\,dt - x(ax^3 + bx^2 + cx + d)$

$= 3\cdot\left(\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right) - ax^4 - bx^3 + cx^2 + dx$

$= -\frac{ax^4}{3} + \frac{cx^2}{2} + 2dx$

$\therefore (Lp)(x) = -\frac{ax^4}{4} + \frac{cx^2}{2} + 2dx \in \mathbb{P}_3(\mathbb{R},\mathbb{R}) \Leftrightarrow a = 0.$

So $p \in Dom(L) \Leftrightarrow p \in \mathbb{P}_2(\mathbb{R},\mathbb{R})$

$\Rightarrow Dom(L) = \mathbb{P}_2(\mathbb{R},\mathbb{R})$

$\{x^2, x, 1\}$ is a basis of Dom(L)

2.b) Find a basis for $Nul(L)$.

$Nul(L) \subseteq Dom(L)$.

$p(x) = ax^2 + bx + c \in Nul(L) \Leftrightarrow (Lp)(x) = 0 \Leftrightarrow 3\cdot\int_0^x (at^2 + bt + c)\,dt - x\cdot(ax^2 + bx + c) = 0$

$\Leftrightarrow 3\cdot\left(\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right) - ax^3 - bx^2 - cx = 0$

$\Leftrightarrow \frac{bx^2}{2} + cx = 0$

$\Leftrightarrow b = c = 0$

$\Rightarrow Nul(L) = \{ax^2 \mid a \in \mathbb{R}\}$

$= span\{x^2\}$

So $\{x^2\}$ is a basis of Nul(L).

2.c) Find a basis for $Ran(L)$.

Extend $\{x^2\}$ to a basis of Dom(L), for example, $\{x^2, x, 1\}$ is a basis of Dom(L)

$\Rightarrow L(\{x, 1\}) = \{L(x), L(1)\} = \left\{\frac{x^2}{2}, 2x\right\}$

$\Rightarrow \{x, x^2\}$ is a basis of Ran(L)

2.d) Adress the existence and uniqueness problems for the linear equation $Lp = q$ with q given by $q(x) := 1 + x^3$.

$1 + x^3 \notin Ran(L)$. since there is no d_1, d_2 s.t. $d_1 x + d_2 x^2 = 1 + x^3$

So $Lp = q$ has no solution

Math 107, Fall 2012, Quiz # 5f

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (8 points)Let $L : \mathbb{C}^2 \to \mathbb{C}^2$ be a linear operator given by $L(w, z) := (w + iz, z - iw)$. Determine the matrix representation of L in the bases (E, F), where $E := \{(1, i), (1, -i)\}$ and $F := \{(1, i), (i, 0)\}$ are different bases of $\mathbb{C}^2$.

Quiz 5c / Problem 1

Problem 2 (12 points) Let $L : \mathbb{C}^2 \to \mathbb{C}^2$ be the every-where defined linear operators defined by $L(w, z) := ((4 + 2i)w - 2iz, (1 + 3i)w + (1 - i)z)$.

2.a) Find a basis for $Nul(L)$.

$Nul(L) = \{(w,z) \mid L(w,z) = (0,0)\}$.

$(w,z) \in Nul(L) \iff L(w,z) = (0,0) \iff \begin{matrix} (4+2i)w - 2iz = 0 \\ (1+3i)w + (1-i)z = 0 \end{matrix} \iff w = \frac{1+2i}{5} z$

So $Nul(L) = \left\{ \left(\frac{1+2i}{5}, 1\right) z \;\middle|\; z \in \mathbb{C} \right\} = \operatorname{span}\left\{ \left(\frac{1+2i}{5}, 1\right) \right\}$.

$\boxed{\text{So } \left\{ \left(\frac{1+2i}{5}, 1\right) \right\} \text{ is a basis of } Nul(L)}$

2.b) Find a basis for $Ran(L)$.

Extend $\left\{ \left(\frac{1+2i}{5}, 1\right) \right\}$ to a basis of $\mathbb{C}^2$. For example, $\left\{ \left(\frac{1+2i}{5}, 1\right), (0,1) \right\}$ is a basis of $\mathbb{C}^2$. (Check that it's a linearly independent set and since $\mathbb{C}^2$ is 2-dimensional, it is a basis of $\mathbb{C}^2$),

So $L(\{(0,1)\}) = \{L(0,1)\} = \{(-2i, 1-i)\}$ is a basis of $Ran(L)$.

2.c) Adress the existence and uniqueness problems for the linear equation $L\vec{x} = (1, 0)$.

Check that whether $(1,0) \in Ran(L)$ or not.

We know that $\{(-2i, 1-i)\}$ is a basis of $Ran(L)$. If $(1,0) \in Ran(L)$, then there would be $\alpha \in \mathbb{C}$ s.t. $\alpha \cdot (-2i, 1-i) = (1,0)$

$\Rightarrow \left.\begin{matrix} -2\alpha i = 1 \Rightarrow \alpha = \frac{1}{2} \\ (1-i)\alpha = 0 \Rightarrow \alpha = 0 \end{matrix}\right\}$ contradiction

So $(1,0) \notin Ran(L)$, that is $L\vec{x} = (1,0)$ has no solution