Math 107, Fall 2012, Quiz # 2a

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (10 points) Determine whether the following series are absolutely convergent, conditionaly convergent, or divergent:

$$(i)\ \sum_{n=2}^{\infty}(-1)^n\frac{1}{ln(n)}$$

$$(ii)\ \sum_{n=1}^{\infty}\frac{sin(4n)}{4^n}$$

i) $\sum_{n=2}^{\infty}(-1)^n\frac{1}{\ln(n)}$ is an alternating series.

As $\ln(n+1) > \ln(n)$ $\forall n\geq 2$, $\frac{1}{\ln(n+1)} < \frac{1}{\ln(n)}$. So $\left\{\frac{1}{\ln(n)}\right\}$ is decreasing. And $\lim_{n\to\infty}\frac{1}{\ln(n)} = 0$. Hence, by Alternating Series Test, $\sum_{n=2}^{\infty}(-1)^n\frac{1}{\ln(n)}$ converges.

Now, $\ln(n) < n$ so $\frac{1}{\ln(n)} > \frac{1}{n}$, and since $\sum_{n=2}^{\infty}\frac{1}{n}$ is divergent, $\sum_{n=2}^{\infty}\frac{1}{\ln(n)}$ diverges by Comparison test.

Thus, $\sum_{n=2}^{\infty}(-1)^n\frac{1}{\ln(n)}$ is conditionally convergent.

ii) $\left|\frac{\sin(4n)}{4^n}\right| \leq \frac{1}{4^n}$ and $\sum_{n=1}^{\infty}\frac{1}{4^n}$ converges since it's a geometric series and $\frac{1}{4}<1$.

So $\sum_{n=1}^{\infty}\frac{\sin(4n)}{4^n}$ converges absolutely by Comparison test.

Problem 2 (10 points)

2.a) (3 points) What is an alternating series. Under which conditions does an alternating series converge?

An alternating series is a series whose terms are alternately positive and negative.

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ satisfies i) $b_{n+1} \leq b_n \quad \forall n \geq 1$

ii) $\lim_{n\to\infty} b_n = 0$

then the series converges.

2.b) (7 points) For what values of p is the following series convergent?

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n+p}$$

Let $b_n = \frac{1}{n+p}$. Then

i) $\frac{1}{n+1+p} < \frac{1}{n+p} \quad \forall n \geq 1$ and for all $p \geq 0$

ii) $\lim_{n\to\infty} \frac{1}{n+p} = 0$ for all $p \geq 0$.

So the series converges (by Alternating Series test) for all $p \geq 0$.

Math 107, Fall 2012, Quiz # 2b

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (3 points) What can you say about the series $\sum a_n$ in each of the following cases?

(i)) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$

Since $0 < 1$ $\sum a_n$ converges by Ratio Test

(ii)) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0.9$

Since $0.9 < 1$, $\sum a_n$ converges by Ratio Test

(ii)) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

Ratio test is inconclusive. We can not say anything about convergence or divergence of $\sum a_n$

Problem 2 (7 points) For which positive integers k is the following series convergent?

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$$

Use the Ratio test:

$$\lim_{n\to\infty} \left| \frac{\frac{((n+1)!)^2}{(k(n+1))!}}{\frac{(n!)^2}{(kn)!}} \right| = \lim_{n\to\infty} \frac{(n+1)^2 \cdot (kn)!}{(kn+k)!}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(kn+k)(kn+k-1)\cdots(kn+1)} =$$

For $k=1$; $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} (n+1) = \infty$

For $k=2$; $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$

For $k \geq 3$, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$

So $\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$ converges for $k \geq 2$ by Ratio Test.

Problem 3 (10 points) Show that if $a_n > 0$ and $\lim_{n\to\infty} na_n \neq 0$, then $\sum a_n$ is divergent.

$$\lim_{n\to\infty} na_n = \lim_{n\to\infty} \frac{a_n}{1/n}.$$

So apply the Limit Comparison test with $b_n = \frac{1}{n}$.

Since $\lim_{n\to\infty} \frac{a_n}{1/n} > 0$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series with $p=1$), $\sum_{n=1}^{\infty} a_n$ is divergent by Limit Comparison test.

Math 107, Fall 2012, Quiz # 2e

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (10 points)

1.a) (3 points) Give the definition of the alternating series. Under which conditions does an alternating series converge?

An alternating series is a series whose terms are alternately positive and negative

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ satisfies i) $b_{n+1} \leq b_n \ \forall n \geq 1$,

ii) $\lim_{n\to\infty} b_n = 0$

then the series converges.

1.b) (7 points) Prove that if $a_n \geq 0$ and $\sum a_n$ converges, then $\sum a_n^2$ also converges.

Since $\sum a_n$ converges, $\lim_{n\to\infty} a_n = 0$ So for $\varepsilon = 1$, there exists N such that $|a_n - 0| < 1$ for all $n \geq N$.

$\Rightarrow 0 \leq a_n < 1$ for all $n \geq N$

$\Rightarrow 0 \leq a_n^2 < a_n$ for all $n \geq N$

So since $\sum a_n$ converges, $\sum a_n^2$ converges by Comparison test.

Problem 2 (10 points) Test the following series for convergence or divergence:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{ln(n+4)}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[4]{n}}$$

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+4)}$ is an alternating series with $b_n = \frac{1}{\ln(n+4)}$

i) Since for all $n \geq 1$, $\ln(n+5) > \ln(n+4)$, $b_{n+1} = \frac{1}{\ln(n+5)} < \frac{1}{\ln(n+4)} = b_n$ $\forall n \geq 1$. So b_n is decreasing

ii) As $0 < \frac{1}{\ln(n+4)} < \frac{1}{n+4}$, $\lim_{n\to\infty} \frac{1}{\ln(n+4)} = 0$ by Squeeze Thm.

So $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+4)}$ converges by Alternating series test.

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[4]{n}}$ is alternating series test with $b_n = \frac{1}{\sqrt[4]{n}}$

i) Since for all $n \geq 1$ $\sqrt[4]{n+1} > \sqrt[4]{n}$, $b_{n+1} = \frac{1}{\sqrt[4]{n+1}} < \frac{1}{\sqrt[4]{n}} = b_n$. So b_n is decreasing

ii) $\lim_{n\to\infty} \frac{1}{\sqrt[4]{n}} = 0$

So $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[4]{n}}$ converges by Alternating series test

Math 107, Fall 2012, Quiz # 2d

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (10 points) Determine whether the following series are absolutely convergent, conditionaly convergent, or divergent.

(i) $\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\cdot \ldots \cdot(2n)}{n!}$

(ii) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$

i) $\frac{2.4.6.\ldots(2n)}{n!} = \frac{2.1.2.2.2.3\ldots 2.n}{n!} = \frac{2^n.n!}{n!} = 2^n$

$\Rightarrow \sum_{n=1}^{\infty} \frac{2.4.6\ldots 2n}{n!} = \sum_{n=1}^{\infty} 2^n$

As $\lim_{n\to\infty} 2^n = \infty$, $\sum_{n=1}^{\infty} \frac{2.4.6\ldots 2n}{n!}$ diverges by Test for Divergence

ii) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ is an alternating series with $b_n = \frac{1}{\ln(n)}$

As $\ln(n+1) > \ln(n)$ for all $n \geqslant 2$, $b_{n+1} = \frac{1}{\ln(n+1)} < \frac{1}{\ln(n)} = b_n$ $\forall n \geqslant 2$. So b_n is decreasin

$\lim_{n\to\infty} \frac{1}{\ln(n)} = 0$.

So by Alternating Series Test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges.

Now, as $\ln(n) < n$ for all $n \geqslant 2$, $\frac{1}{\ln(n)} > \frac{1}{n}$. As $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges (p-series with p=1)

$\sum_{n=1}^{\infty} \frac{1}{\ln(n)}$ diverges by Comparison Test.

Hence $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n)}$ is conditionally convergent.

Problem 2 (10 points) Show that the series is convergent. How many terms of the series do we need to add in order to find sum to the indicated accuracy?

$$\sum_{n=1}^{\infty}(-1)^{n-1}n2^{-n} \quad (|\text{error}\,| < 0.01)$$

$\sum_{n=1}^{\infty}(-1)^{n-1}n2^{-n}$ is an alternating series with $b_n = \frac{n}{2^n}$.

(i) $\{b_n\}$ is decreasing for $n \geq 2$ since

$$\left(\frac{x}{2^x}\right)' = \frac{1\cdot 2^x - x\,2^x \ln 2}{2^{x+1}} = \frac{2^x(1-x\ln 2)}{2^{x+1}} < 0 \text{ for } 1 - x\ln 2 < 0 \Rightarrow x > \frac{1}{\ln 2} \approx 1.4$$

(ii) $\lim_{n\to\infty} \frac{n}{2^n} \overset{\text{l'H}}{=} \lim_{n\to\infty} \frac{1}{2^n \ln 2} = 0.$

Thus, by Alternating Series Test, $\sum_{n=1}^{\infty}(-1)^{n-1}n2^{-n}$ converge.

Now, by Alternating Series Estimation Thm, we have

$$|R_n| \leq b_{n+1}$$

So it's enough to find n s.t. $b_{n+1} < 0.01$.

Notice that $b_{10} = \frac{10}{2^{10}} = \frac{10}{1024} < 0.01$.

$\Rightarrow |R_9| < 0.01$.

So we need to add 9 terms of the series.

Math 107, Fall 2012, Quiz # 2e

You have 25 minutes.

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☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (8 points) Prove that if $a_n \geq 0$ and $\sum a_n$ converges, then $\sum a_n^2$ also converges.

Since $\sum a_n$ converges, $\lim_{n\to\infty} a_n = 0$. So for $\varepsilon = 1$, there exists N such that $|a_n - 0| < 1$ for all $n \geq N$.

$\Rightarrow 0 \leq a_n < 1$ for all $n \geq N$

$\Rightarrow 0 \leq a_n^2 < a_n$ for all $n \geq N$

So since $\sum a_n$ converges, $\sum a_n^2$ converges by Comparison Test.

Problem 2 (12 points) Determine whether the following series converge or diverge:

(i) $\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n})}$

(ii) $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$

(iii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[4]{n}}$

i) $a_n = \frac{n-1}{n^2\sqrt{n}} = \frac{n-1}{n^{5/2}} < \frac{n}{n^{5/2}} = \frac{1}{n^{3/2}} = b_n \quad \forall n \geq 1$

As $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (p-series with $p = 3/2 > 1$), $\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$ converges by Comparison Test

ii) Apply the limit comparison test with $b_n = \frac{1}{n}$.

$\lim_{n\to\infty} \frac{\frac{e^{1/n}}{n}}{\frac{1}{n}} = \lim_{n\to\infty} e^{1/n} = 1 > 0$ $\left[\begin{array}{l} a_n = 1/n, \ \lim_{n\to\infty} 1/n = 0 \text{ and } f(x) = e^x \text{ is cont. at } 0 \\ \Rightarrow \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} e^{1/n} = f(0) = 1 \end{array} \right]$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges [p-series with $p = 1$], $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$ diverges by Limit Comp. Test

iii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[4]{n}}$ is an alternating series with $b_n = \frac{1}{\sqrt[4]{n}}$.

As $\sqrt[4]{n+1} > \sqrt[4]{n}$ for all $n \geq 1$, $b_{n+1} = \frac{1}{\sqrt[4]{n+1}} < \frac{1}{\sqrt[4]{n}} = b_n$ for all $n \geq 1$. So b_n is decreasing

$\lim_{n\to\infty} \frac{1}{\sqrt[4]{n}} = 0$

So $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[4]{n}}$ converges by Alternating Series Test.

Math 107, Fall 2012, Quiz # 2f

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (10 points) Test the series for convergence or divergence:

$$(i)\ \sum_{n=1}^{\infty}(-1)^n \frac{n}{10^n}$$

$$(ii)\ \sum_{n=1}^{\infty}\frac{sin(4n)}{4^n}$$

i) $\sum_{n=1}^{\infty}(-1)^n \frac{n}{10^n}$ is an alternating series with $b_n = \frac{n}{10^n}$

$\{b_n\}$ is decreasing for $n \geq 1$, since $\left(\frac{x}{10^x}\right)' = \frac{1 \cdot 10^x - x 10^x \ln(10)}{(10^x)^2} = \frac{10^x(1 - x\ln(10))}{(10^x)^2} < 0$

for $1 - x\ln(10) < 0$, i.e, $x > \frac{1}{\ln(10)} \approx 0.4$

And $\lim_{n\to\infty} \frac{n}{10^n} \overset{L'H}{=} \lim_{n\to\infty} \frac{1}{10^n \ln(10)} = 0$

So the series converges by Alternating series test.

ii) Apply Comparison Test with $b_n = \frac{1}{4^n}$

$\left|\frac{\sin(4n)}{4^n}\right| \leq \frac{1}{4^n}$ as $|\sin(4n)| \leq 1$ $\forall n \geq 1$.

Since $\sum_{n=1}^{\infty} \frac{1}{4^n}$ converges (geometric series with $\frac{1}{4} < 1$), $\sum_{n=1}^{\infty} \frac{|\sin(4n)|}{4^n}$ converges

As absolute convergence of a series implies convergence of a series, $\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$ is convergent.

Problem 2 (10 points) Show that if $a_n > 0$ and $\lim_{n\to\infty} na_n \neq 0$, then $\sum a_n$ is divergent.

$$\lim_{n\to\infty} na_n = \lim_{n\to\infty} \frac{a_n}{1/n}.$$

So apply Limit Comparison test with $b_n = \frac{1}{n}$

Since $\lim_{n\to\infty} \frac{a_n}{1/n} > 0$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series with $p \leq 1$),

$\sum_{n=1}^{\infty} a_n$ is divergent by Limit Comparison Test.