Math 107, Fall 2012, Quiz # 1a

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (3 points) Give the definition of the following terms.

1.a) A real sequence: is a function $f: \mathbb{N} \to \mathbb{R}$ where $\mathbb{N} = \{1, 2, \dots\}$

1.b) A decreasing sequence: is a sequence a_n such that $a_n \geq a_{n+1}$ for all n.

1.c) Domain of a function: Let $f: A \to B$ be a function. Then domain of A is the subset of A which consists of elements $a \in A$ such that $f(a)$ is defined as an element in B.

Problem 2 (7 points) Determine whether the following series convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$$

First note that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is divergent by p-test.

Now we will apply $\boxed{\text{limit comparison test}}$:

$$a_n := \frac{1}{\sqrt{n+4}}, \quad b_n := \frac{1}{\sqrt{n}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+4}} = 1$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$ is also divergent

Let $L := \lim_{n\to\infty} a_n$. Since $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$, taking the limit, we get from the equality

$$a_{n+1} = \frac{1}{3-a_n}$$

that $L = \frac{1}{3-L}$. So $L^2 - 3L + 1 = 0$. Solutions are given by

$$L_{1,2} = \frac{3 \pm \sqrt{5}}{2}$$

Since $\frac{3+\sqrt{5}}{2} > 2$, $L = \frac{3-\sqrt{5}}{2}$

Problem 3 (10 points) Show that the sequence defined by

$$a_1 = 2, \quad a_{n+1} = \frac{1}{3 - a_n}$$

is bounded and decreasing. Also decide whether the ~~series~~ sequence is convergent or divergent. If it is convergent, find its limit.

a_n is decreasing

Using induction we will show that the sequence is decreasing. So our claim is $\forall n \in \mathbb{N}$. $a_n \geq a_{n+1}$.

(i) For $n = 1$ we have

$a_1 = 2$ and $a_2 = \frac{1}{2}$

which implies that $a_1 \geq a_2$.

(ii) Now suppose that $a_{n-1} \geq a_n$ for some $n \in \mathbb{N}$.
We will show that this implies $a_n \geq a_{n+1}$.

$$a_{n+1} = \frac{1}{3-a_n} \leq \frac{1}{3-a_{n-1}} = a_n$$

the induction hypothesis $a_{n-1} \geq a_n$ implies that $-a_n \geq -a_{n-1}$

Hence, the induction is complete and the sequence is decreasing.

a_n is bounded

Now we will show that a_n is bounded. First since a_n is decreasing a_n is bounded from above by 2.
So $a_n \leq 2$ for all n. This implies that
$3 - a_n \geq 1$ for all n. Hence $\frac{1}{3-a_n} > 0$ for all n
which implies that a_n is bounded from below by 0.

Since a_n is decreasing and bounded, by monotone sequence theorem a_n is convergent. (Continues from the top)

Math 107, Fall 2012, Quiz # 1b

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (3 points) Give the definition of the following terms.

1.a) Onto function: Let $f: A \to B$ be a function. Then f is said to be onto if for all $b \in B$ there exist an element $a \in A$ with $f(a) = b$

1.b) A sequence, that is bounded below: is a sequence a_n such that there exist a real number M satisfying $a_n \geq M$ for all n.

1.c) Geometric series: is a series of the form $\sum_{n=1}^{\infty} r^n$ where r is a real number

Problem 2 (7 points) Show that the sequence

$$\{\frac{e^n + e^{-n}}{e^{2n} - 1}\}$$

is convergent and find its limit.

Dividing each term with e^{2n} we get

$$\frac{e^{-n} + e^{-3n}}{1 - e^{-2n}}$$

Since $\lim_{n \to \infty} e^{-an} = 0$ for all $a > 0$, we conclude that the sequence converges to $\frac{0}{1} = 0$.

Now, if $\sum_{n=1}^{\infty}\left(\frac{1}{2^n+1}+\frac{2}{n}\right)$ were convergent, then, since the difference of convergent series is convergent, we would have $\sum_{n=1}^{\infty}\left(\frac{1}{2^n+1}+\frac{2}{n}\right)-\sum_{n=1}^{\infty}\frac{1}{2^n+1}=\sum_{n=1}^{\infty}\frac{2}{n}$ convergent, which is not true. So our series is divergent.

Problem 3 (10 points) Determine whether the following series are convergent or divergent. If convergent find its sum. (Give reasons for your answer.)

$$(i)\ \sum_{n=1}^{\infty} ln\left(\frac{n^2+1}{2n^2+1}\right)$$

$$(ii)\ \sum_{n=1}^{\infty}\left(\frac{1}{2^n+1}+\frac{2}{n}\right)$$

(i) Note that

$$\lim_{n\to\infty}\left(\frac{n^2+1}{2n^2+1}\right)=\frac{1}{2}$$

Since $\ln x$ is a continuous function, we have

$$\lim_{n\to\infty}\ln\left(\frac{n^2+1}{2n^2+1}\right)=\ln\left(\lim_{n\to\infty}\frac{n^2+1}{2n^2+1}\right)$$

$$=\ln\left(\frac{1}{2}\right)\neq 0$$

Hence, the series is divergent as the general term does not converge to 0.

(ii) First, observe that

$$\frac{1}{2^n+1}<\frac{1}{2^n}$$

Since $\sum_{n=1}^{\infty}\frac{1}{2^n}$ is a geometric series with $\frac{1}{2}<1$ it is convergent. So by comparison $\sum_{n=1}^{\infty}\frac{1}{2^n+1}$ is convergent.

Also if $\sum_{n=1}^{\infty}\frac{2}{n}$ were convergent, then

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{2}{n}=\sum_{n=1}^{\infty}\frac{1}{n}$$

would be convergent which is not true. (harmonic series)

(Continues from the top of the page.)

Math 107, Fall 2012, Quiz # 1c

You have 25 minutes.

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☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (3 points) Give the definition of the following terms.

1.a) Equal sets: Two sets A, B are said to be equal if $A \subseteq B$ and $B \subseteq A$

1.b) A decreasing sequence: is a sequence such that $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

1.c) Geometric series: is a series of the form $\sum_{n=1}^{\infty} r^n$ where r is a real number.

Problem 2 (7 points) Determine whether the following sequence is increasing, decreasing or not monotonic. Is the sequence bounded?

$$a_n = \frac{2n-3}{3n+4}$$

Quiz 1f, problem 3, part 2

Problem 3 (10 points) Find the values of x for which the series converges. Find the sum of series for those values of x.

$$(i)\ \sum_{n=0}^{\infty} \frac{cos^n x}{2^n}$$

$$(ii)\ \sum_{n=0}^{\infty} \frac{(x+3)^n}{2^n}$$

i) Note that for each $x \in \mathbb{R}$ the series $\sum_{n=0}^{\infty} \frac{\cos^n x}{2^n} = \sum_{n=0}^{\infty} \left(\frac{\cos x}{2}\right)^n$

is a geometric series and the series converges for all x satisfying

$$\left|\frac{\cos x}{2}\right| < 1$$

which is true for all $x \in \mathbb{R}$. Hence the series is convergent for all $x \in \mathbb{R}$ and converges to $\frac{1}{1-\frac{\cos x}{2}} = \frac{2}{2-\cos x}$

ii) By the same reasoning in part (i) the series converges for all $x \in \mathbb{R}$ satisfying

$$\left|\frac{x+3}{2}\right| < 1 \Rightarrow -2 < x+3 < 2$$

Hence the series converges for $x \in (-5, -1)$ to the number $\frac{1}{1-\frac{x+3}{2}} = -\frac{2}{x+1}$

Math 107, Fall 2012, Quiz # 1d

You have 25 minutes.

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☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (3 points) Give the definition of the following terms.

1.a) Equal sets : Let A, B be two sets. Then A is equal to be if $A \subseteq B$ and $B \subseteq A$.

1.b) A sequence, that is bounded below: is a sequence a_n such that there exist a real number M satisfying $a_n \geq M$ for all n.

1.c) A convergent series: is a series whose partial sum sequence is convergent.

Problem 2 (7 points)Determine whether the following sequence is increasing, decreasing or not monotonic. (Give reasons for your answers.)

$$a_n = n + \frac{1}{n}$$

$$a_{n+1} - a_n = 1 + \frac{1}{n+1} - \frac{1}{n} > 0$$

So a_n is increasing.

Since a_n is increasing and bounded, monotone sequence theorem implies that a_n is convergent.

Problem 3 (10 points) A sequence is given by $a_1 = \sqrt{6}$, $a_{n+1} = \sqrt{6 + a_n}$. By induction or otherwise, show that $\{a_n\}$ is increasing and bounded by 4. Apply the Monotonic Sequence Theorem to show $lim_{n \to \infty} a_n$ exists.

a_n is increasing

By induction, we will show that for all $n \geq 1$

$$a_{n+1} \geq a_n$$

i) For $n=1$:

$$a_2 = \sqrt{6+\sqrt{6}} > \sqrt{6} = a_1$$

ii) Suppose that $a_n \geq a_{n-1}$. We will prove that this implies $a_{n+1} \geq a_n$ and the induction will be complete.

$$a_{n+1} = \sqrt{6+\sqrt{a_n}}$$

$$a_n = \sqrt{6+\sqrt{a_{n-1}}}$$

Note that, the induction hypothesis $a_n \geq a_{n-1}$ implies that $\sqrt{a_n} \geq \sqrt{a_{n-1}}$. Hence

$$a_{n+1} \geq a_n$$

a_n is bounded

Now using induction again, we will show that a_n is bounded from above by 4. So our claim is that $a_n \leq 4$ for all n.

(i) For $n=1$, $a_1 = \sqrt{6} < 4$

(ii) Assume that $a_n \leq 4$ for some n.

Then $a_{n+1} = \sqrt{6+\sqrt{a_n}} \leq \sqrt{6+2} = \sqrt{8} < 4$.

Hence the induction is complete. (Continues from the top of the page)

Math 107, Fall 2012, Quiz # 1e

You have 25 minutes.

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☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (3 points) Give the definition of the following terms.

1.a) Domain of a function: Let $f: A \to B$ be a function. Then domain of A is the subset of A which consists of elements $a \in A$ such that $f(a)$ is defined as an element in B.

1.b) Convergent series: is a series whose partial sum sequence converges.

1.c) A sequence, that is bounded below: is a sequence a_n such that there exist a real number M satisfying $a_n \geq M$ for all n.

Problem 2 (7 points) Detemine whether the following series is convergent or divergent. If it is convergent find its sum.

$$\sum_{n=1}^{\infty} \frac{n(n+2)}{(n+2)^2}$$

We have

$$\lim_{n\to\infty} \frac{n(n+2)}{(n+2)^2} = \lim_{n\to\infty} \frac{n^2+2n}{n^2+2n+2} = 1.$$

So the series is divergent.

Problem 3 (10 points) The Fibonacci sequence f_n is defined as $f_1 = 1,\ f_2 = 1$ and $f_n = f_{n-1}+f_{n-2}$ for $n \geq 3$. Let us define another sequence using the Fibonacci sequence: $a_n = \frac{f_{n+1}}{f_n}$.
Show that $a_{n-1} = 1 + 1/a_{n-2}$. Assuming that $\{a_n\}$ is convergent, find its limit.

We have

$$a_{n-1} = \frac{f_n}{f_{n-1}} \quad \text{and} \quad 1 + \frac{1}{a_{n-2}} = 1 + \frac{f_{n-2}}{f_{n-1}}$$

Hence

$$a_{n-1} - \left(1 + \frac{1}{a_{n-2}}\right) = \frac{f_n - f_{n-2}}{f_{n-1}} - 1$$

$$= \frac{f_n - f_{n-1} - f_{n-2}}{f_{n-1}}$$

By definition of f_n, for $n \geq 3$

$$f_n - f_{n-1} - f_{n-2} = 0.$$

Hence, for all $n \geq 1$

$$\boxed{a_{n-1} = 1 + \frac{1}{a_{n-2}}}$$

Now suppose that a_n is convergent and

$$\lim_{n\to\infty} a_n = L.$$

Then taking the limit in the following equality

$$a_{n-1}a_{n-2} = a_{n-2} + 1$$

we get that $L^2 = L + 1$, (Note that $\lim_{n\to\infty} a_{n-2} = \lim_{n\to\infty} a_{n-1} = \lim_{n\to\infty} a_n$)

Then $L_{1,2} = \frac{1 \pm \sqrt{5}}{2}$. Since $a_n > 0$ $\boxed{L = \frac{1+\sqrt{5}}{2}}$

Now let $p>1$. Then we can again apply integral test if we start the series from an integer n such that $\ln n>1$. Then

$$\int_3^\infty \frac{1}{x(\ln x)^p}dx = \int_{\ln 3}^\infty \frac{du}{u^p} = \frac{u^{-p+1}}{1-p}\Big|_{\ln 3}^\infty$$

. Since $p>1$, $1-p<0$ hence the integral is finite which implies the convergence for $p>1$.

Math 107, Fall 2012, Quiz # 1f

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☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (3 points) Give the definition of the following terms.

1.a) A real sequence: is a function $f: \mathbb{N} \to \mathbb{R}$ where $\mathbb{N} := \{1, 2, 3, \dots\}$

1.b) Onto function: is a function $f: A \to B$ such that for every element $b \in B$ there is an element $a \in A$ satisfying $f(a) = b$

1.c) Geometric series: is a series of the form $\sum_{n=1} r^n$ where r is a real number.

Problem 2 (7 points) Find the values of p for which the series is convergent:

$$\sum_{n=2}^{\infty} \frac{1}{n(ln\ n)^p}$$

First we will analyze the case $p=1$.

Then since $\frac{1}{x \ln x}$ is a decreasing, positive, continuous function,

we can apply the integral test.

$$\int_2^\infty \frac{1}{x \ln x}dx = \int_{\ln 2}^\infty \frac{du}{u} = \infty.$$

$u = \ln x$

So $\sum_{n=2}^\infty \frac{1}{n \ln n}$ is divergent.

Since for $p<1$ $(\ln n)^p < \ln n$ (here we take $n>3$ so that $\ln n > 1$)

we have $\frac{1}{n(\ln n)^p} > \frac{1}{n \ln n}$. So by comparison test the series diverges also for $p<1$. Note that for $p<0$

$\frac{1}{n(\ln n)^p}$ need not be decreasing, so we can't apply integral test. (if decreasing we have to show)

Problem 3(10 points) Determine whether the following sequences are increasing, decreasing or not monotonic. Are they bounded? (Give reasons for your answers.)

$$(i)\ a_n = n + \frac{1}{n}$$

$$(ii)\ a_n = \frac{2n-3}{3n+4}$$

i) $a_n = n + \frac{1}{n}$

$a_{n+1} - a_n = 1 + \frac{1}{n+1} - \frac{1}{n} > 0$

So for all n. $a_{n+1} > a_n$ which implies that a_n is increasing. Since a_n is a positive sequence it is bounded from below by 0. Also a_n diverges to infinity, so it is not bounded from above.

Hence a_n is not bounded

ii) $a_n = \frac{2n-3}{3n+4}$

Calculating $a_{n+1} - a_n$ we get

$a_{n+1} - a_n = \frac{17}{(3n+7)(3n+4)} > 0$

So a_n is increasing.

Also

$\lim_{n \to \infty} a_n = \frac{2}{3}$

Hence a_n is convergent which implies that a_n is bounded.