Solutions to Math 107, Midterm Exam 2 Problems.

Problem 1 (10 points) Give the definition of the following terms.

- modulus (or absolute value) of a complex number z:

This is the real number defined by

$$|z| := \sqrt{Re(z)^2 + Im(z)^2}$$

- span of a subset of a vector space: This is the set consisting of all the linear combinations of the subset.

- basis of a vector space: This is a linearly-independent subset that spans the vector space.

- isomorphism: An isomorphism is a linear bijection that maps a vector space to a vector space, i.e., it is an everywhere-defined, onto, and one-to-one linear operator $\phi : V \to W$ where V and W are vector spaces over $\mathbb{R}$ or $\mathbb{C}$.

- Hermitian matrix: A matrix $\mathbb{H}$ is called Hermitian if it equals its adjoint, i.e. $\mathbb{H} = \mathbb{H}^{\dagger}$ where $\mathbb{H}^{\dagger} := \overline{\mathbb{H}}^{T}$!

Problem 2 Let $\mathfrak{M}(2,2;\mathbb{R})$ be the vector space of 2×2 real matrices and U be the subset of $\mathfrak{M}(2,2;\mathbb{R})$ consisting of matrices $\mathbf{L} := [L_{ij}]$ such that the sum of the entries of $\mathbf{L}$ is zero, i.e., $L_{11} + L_{12} + L_{21} + L_{22} = 0$.

2.a (6 points) Show that U is a subspace of $\mathfrak{M}(2,2;\mathbb{R})$.

$$U := \{ [L_{ij}] \in \mathfrak{M}(2,2;\mathbb{R}) \mid \underbrace{L_{11}+L_{12}+L_{21}+L_{22} = 0}_{\sum_{i=1}^{2}\sum_{j=1}^{2} L_{ij} = 0 \quad (*)} \}$$

1) clearly $\mathbb{O} := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ satisfies $(*)$

So $\mathbb{O} \in U$

2) $\forall\, \mathbb{L}, \mathbb{J} \in U$, $\forall \alpha, \beta \in \mathbb{R}$,

$$\sum_{i=1}^{2}\sum_{j=1}^{2} L_{ij} = 0 \quad \& \quad \sum_{i=1}^{2}\sum_{j=1}^{2} J_{ij} = 0 \Rightarrow$$

Let $\mathbb{M} := \alpha \mathbb{L} + \beta \mathbb{J} \Rightarrow M_{ij} = \alpha L_{ij} + \beta J_{ij}$

$$\Rightarrow \sum_{i=1}^{2}\sum_{j=1}^{2} M_{ij} = \sum_{i=1}^{2}\sum_{j=1}^{2} (\alpha L_{ij} + \beta J_{ij})$$

$$= \alpha \underbrace{\sum_{i=1}^{2}\sum_{j=1}^{2} L_{ij}}_{0} + \beta \underbrace{\sum_{i=1}^{2}\sum_{j=1}^{2} J_{ij}}_{0}$$

$$= 0$$

$\Rightarrow \mathbb{M} \in U$

(1) & (2) $\Rightarrow U$ is a subspace of $\mathfrak{M}(2,2;\mathbb{R})$.

2.b (4 points) Find a basis for U and determine its dimension (You do not need to prove that the set you claim to be a basis is actually a basis.)

$\forall \mathbb{L} \in U$, $L_{11} + L_{12} + L_{21} + L_{22} = 0 \Rightarrow L_{22} = -(L_{11} + L_{12} + L_{21})$

$$\Rightarrow \mathbb{L} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & -L_{11}-L_{12}-L_{21} \end{bmatrix} = L_{11}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + L_{12}\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + L_{21}\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

$\mathcal{B} := \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$ is a basis.

Therefore, $\dim U = 3$.

Problem 3 (10 points) Let V and W be vector spaces and $L : V \to W$ be a linear operator. Using only the definition of a linear operator and its range, show that the range of L is a subspace of W.

$$\mathrm{Ran}(L) := \{ w \in W \mid \exists v \in V,\ w = Lv \}$$

1) $L0 = 0 \Rightarrow 0 \in \mathrm{Ran}(L)$

2) $\forall w_1, w_2 \in \mathrm{Ran}(L)$, let $v_1, v_2 \in V$ such that

$Lv_1 = w_1$ and $Lv_2 = w_2$ (*)

$\Rightarrow \forall \alpha_1, \alpha_2 \in \mathbb{F}$,

$$\alpha_1 w_1 + \alpha_2 w_2 = \alpha_1 L v_1 + \alpha_2 L v_2$$
$$= L(\alpha_1 v_1 + \alpha_2 v_2)$$

Clearly $\alpha_1 v_1 + \alpha_2 v_2 \in V \underset{(*)}{\Rightarrow} \alpha_1 w_1 + \alpha_2 w_2 \in \mathrm{Ran}(L)$

① & ② $\Rightarrow$ $\mathrm{Ran}(L)$ is a subspace of W.

Problem 4 Let $B := \{(1, i, 0), (0, 2, 0), (i, 0, 1)\}$.

4.a (10 points) Show that B is a basis of $\mathbb{C}^3$. $\forall \alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$,

$$\alpha_1 (1, i, 0) + \alpha_2 (0, 2, 0) + \alpha_3 (i, 0, 1) = (\underbrace{\alpha_1 + i\alpha_3}_{\beta_1}, \underbrace{i\alpha_1 + 2\alpha_2}_{\beta_2}, \underbrace{\alpha_3}_{\beta_3})$$

$$\Rightarrow \quad \alpha_1 + i\alpha_3 = \beta_1 \Rightarrow \boxed{\alpha_1 = \beta_1 - i\beta_3}$$

$$i\alpha_1 + 2\alpha_2 = \beta_2$$

$$\boxed{\alpha_3 = \beta_3} \qquad \alpha_2 = \frac{1}{2}[\beta_2 - i(\beta_1 - i\beta_3)]$$

$$\Rightarrow \boxed{\alpha_2 = \frac{1}{2}(\beta_2 - i\beta_1 - \beta_3)}$$

Therefore $\forall (\beta_1, \beta_2, \beta_3) \in \mathbb{C}^3$ if we take $\alpha_1 := \beta_1 - i\beta_3$, $\alpha_2 := \frac{1}{2}(\beta_2 - i\beta_1 - \beta_3)$ and $\alpha_3 := \beta_3$ we have $(\beta_1, \beta_2, \beta_3) = \alpha_1 (1, i, 0) + \alpha_2 (0, 2, 0) + \alpha_3 (i, 0, 1) \Rightarrow \langle B \rangle = \mathbb{C}^3$ ①

Next we prove that B is linearly-independent.

Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ such that $\alpha_1 (1, i, 0) + \alpha_2 (0, 2, 0) + \alpha_3 (i, 0, 1) = (0, 0, 0)$

This corresponds to the choice $\beta_1 = \beta_2 = \beta_3 = 0$ in the above formulas which implies $\alpha_1 = \alpha_2 = \alpha_3 = 0$

$\Rightarrow B$ is linearly-independent ②

① & ② $\Rightarrow B$ is a basis of $\mathbb{C}^3$.

4.b (5 points) Find the matrix representation of $(1, 0, 0)$ in B.

$$\alpha_1 (1, i, 0) + \alpha_2 (0, 2, 0) + \alpha_3 (i, 0, 1) = (1, 0, 0)$$

$$\Downarrow$$

$$\alpha_1 = 1 - 0 = 1$$

$$\alpha_2 = \frac{1}{2}(-i) = -\frac{i}{2}$$

$$\alpha_3 = 0$$

$\Rightarrow$ $(1, 0, 0)$ is represented in B by $\begin{bmatrix} 1 \\ -\frac{i}{2} \\ 0 \end{bmatrix}$.

Problem 5 Let $B := \{(1, i, 0), (0, 2, 0), (i, 0, 1)\}$ be as in Problem 4, $E_2 := \{(1, 0), (0, 1)\}$, and $L : \mathbb{C}^3 \to \mathbb{C}^2$ be the linear operator defined by $L(\alpha, \beta, \gamma) := (\alpha - \beta, \beta - \alpha)$.

5.a (10 points) Find the matrix representation of L in the bases (B, E_2).

$$L(1, i, 0) = (1-i, i-1) = (1-i)(1,0) + (i-1)(0,1)$$

$$L(0,2,0) = (-2, 2) = -2(1,0) + 2(0,1)$$

$$L(i,0,1) = (i, -i) = i(1,0) - i(0,1)$$

$$\Rightarrow \quad \mathbb{L} = \begin{bmatrix} 1-i & -2 & i \\ i-1 & 2 & -i \end{bmatrix}$$

5.b (10 points) Find a basis for the null space of L.

$$\forall (\alpha, \beta, \gamma) \in \mathbb{C}^3, \quad \underset{(\alpha-\beta,\ \beta-\alpha)}{\overset{\parallel}{L(\alpha,\beta,\gamma)}} = (0,0) \Rightarrow \alpha - \beta = 0 \Rightarrow \boxed{\alpha = \beta}$$

$$\Rightarrow (\alpha, \beta, \gamma) = (\alpha, \alpha, \gamma) = \alpha(1,1,0) + \gamma(0,0,1)$$

$\Rightarrow$ $B_0 := \{(1,1,0), (0,0,1)\}$ is a basis for $Nul(L)$.

5.c (5 points) Find a basis for the range of L.

$\{(1,1,0), (0,0,1), (1,0,0)\}$ is a basis for $\mathbb{C}^3$

and $\{(1,1,0), (0,0,1)\}$ " " " " $Nul(L)$

$\Downarrow$

$\{L(1,0,0)\}$ is a basis for $Ran(L)$

$\quad \|$

$\{(1,-1)\}$.

Problem 6 (5 points) Let $m \in \mathbb{Z}^+$ and $L : \mathbb{R}^7 \to \mathbb{R}^m$ be an everywhere-defined linear operator. Show that L cannot be invertible, if $m < 7$.

Hint: Think of how the dimension of the null space and range of L are related.

We know that $\dim Nul(L) + \dim Ran(L) = \underbrace{\dim Dom L}_{= 7}$

If $m < 7 \Rightarrow \dim(Ran L) \leq m < 7$

$\Downarrow$

$\dim Nul(L) \geq 1$

$\Rightarrow Nul(L)$ is nontrivial $\Rightarrow L$ is not 1-to-1

$\Downarrow$

L is not invertible.

Problem 7 Let $\mathbf{M} := \begin{bmatrix} 1 & -\imath & 0 \\ 0 & 1+\imath & 1 \\ \imath & -2 & 0 \end{bmatrix}$.

7.a (5 points) Show that $\mathbf{M}$ is an invertible matrix.

$$\det \mathbb{M} = (1)(1+i)(0) + (-i)(i)(1) + (0)(0)(-2)$$
$$- [(0)(1+i)(i) + (-i)(0)(0) + (1)(1)(-2)]$$
$$= 1 - (-2)$$
$$= 3 \neq 0 \Rightarrow \mathbb{M} \text{ is invertible.}$$

7.b (10 points) Find $\mathbf{M}^{-1}$ and show that $\mathbf{M}\mathbf{M}^{-1} = \mathbf{I}$.

$$\mathbb{M}^{-1} = \frac{1}{3} \begin{bmatrix} +2 & -(-i) & -i(1+i) \\ -0 & 0 & -(-2-1) \\ -i & -1 & 1+i \end{bmatrix}^T$$

$$= \frac{1}{3} \begin{bmatrix} 2 & i & 1-i \\ 0 & 0 & 3 \\ -i & -1 & 1+i \end{bmatrix}^T = \frac{1}{3} \begin{bmatrix} 2 & 0 & -i \\ i & 0 & -1 \\ 1-i & 3 & 1+i \end{bmatrix}$$

$$\mathbb{M}\mathbb{M}^{-1} = \begin{bmatrix} 1 & -i & 0 \\ 0 & 1+i & 1 \\ i & -2 & 0 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & 0 & -i \\ i & 0 & -1 \\ 1-i & 3 & 1+i \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2+1 & 0 & -i+i \\ i-1+1-i & 3 & -1-i+1+i \\ 2i-2i & 0 & 1+2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbb{I}$$

Problem 8 (5 points) Show that for every matrix **L**, the condition that the trace of $\mathbf{L}^\dagger\mathbf{L}$ is zero (i.e., $\mathrm{tr}(\mathbf{L}^\dagger\mathbf{L}) = 0$) implies that **L** must be a zero matrix.

$\mathbb{L}^+ = \bar{\mathbb{L}}^T$. If $\mathbb{L}$ is an $m\times n$ matrix, $\mathbb{L}^+$ will be an $n\times m$ matrix and $\mathbb{L}^+\mathbb{L}$ will be an $n\times n$ matrix with entries:

$$(\mathbb{L}^+\mathbb{L})_{ij} = \sum_{k=1}^{m} L^+_{ik} L_{kj} = \sum_{k=1}^{m} \bar{L}_{ki} L_{kj}$$

$$\Rightarrow \mathrm{tr}(\mathbb{L}^+\mathbb{L}) = \sum_{i=1}^{n} (\mathbb{L}^+\mathbb{L})_{ii}$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{m} \bar{L}_{ki} L_{ki} = \sum_{i=1}^{n}\sum_{k=1}^{m} |L_{ki}|^2$$

Therefore $\mathrm{tr}(\mathbb{L}^+\mathbb{L}) = 0 \Rightarrow \sum_{i=1}^{n}\sum_{k=1}^{m} |L_{ki}|^2 = 0$

But $|L_{ki}|^2$ are real and non-negative

$|L_{ki}|^2 = 0$ for all $k \in \{1,2,\dots,m\}$ and all $i \in \{1,2,\dots,n\}$

$\Downarrow$

$L_{ki} = 0$

$\Downarrow$

$\mathbb{L} = \mathbb{O}$.