Solutions to Math 107, Fall 2012 Midterm Exam 1 Problems

Problem 1.a (4 points) Give the definition of the following terms.

- Bounded sequence: A sequence $\{a_n\}$ is bounded if there is $M \in \mathbb{R}^+$ such that $\forall n \in \mathbb{Z}^+$, $|a_n| \leq M$.

- n-degree Taylor polynomial of a function f: $T_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$

Problem 1.b (6 points) Give the statement of the following.

- Monotone Convergence Theorem: Every bounded monotonic sequence converges.

- Limit Comparison Test: Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be a pair series with positive terms such that $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is a positive number. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if so does $\sum_{n=1}^{\infty} b_n$.

- Alternating Series Test: An alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ with $b_n > 0$ for all n converges if $\{b_n\}$ is a decreasing sequence that converges to zero.

Problem 2 (20 points) Prove the following statements.

2.a) The geometric series $\sum_{n=0}^{\infty} r^n$ converges to $\frac{1}{1-r}$ for all $r \in (-1, 1)$.

$$\lim_{N\to\infty} \sum_{n=0}^{N} r^n = \lim_{N\to\infty} (1 + r + r^2 + \dots + r^N)$$

$$= \lim_{N\to\infty} \frac{1 - r^{N+1}}{1-r} \qquad \text{for } r \neq 1$$

$$= \frac{1}{1-r} - \frac{1}{1-r} \underbrace{\lim_{N\to\infty} r^{N+1}}_{=0 \text{ because } |r|<1}$$

$$= \frac{1}{1-r}$$

2.b) If a series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it is a convergent sequence.

Suppose that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Then $\sum_{n=1}^{\infty} |a_n|$ conerges (1).

Clearly for all n $\quad -|a_n| \leq a_n \leq |a_n|$

$$\Rightarrow \quad 0 \leq \underbrace{a_n + |a_n|}_{b_n} \leq 2|a_n| \qquad (2)$$

let $b_n := a_n + |a_n|$

$$(1) \Rightarrow \quad 2\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} 2|a_n| \text{ conergs} \qquad (3)$$

$$(2) \,\&\, (3) \Rightarrow \quad \sum_{n=1}^{\infty} b_n \text{ convergs by comparison test.} \quad (4)$$

$$(1) \,\&\, (4) \Rightarrow \quad \underbrace{\sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} |a_n|}_{\sum_{n=1}^{\infty} b_n - |a_n| \,=\, \sum_{n=1}^{\infty} a_n} \text{ convergs}$$

So $\sum_{n=1}^{\infty} a_n$ converges.

Problem 3 (15 **points**) Determine whether the following series converge. You must justify your response by offering a valid argument.

3.a) $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{\ln(n+2)}$

$$\lim_{n\to\infty} \frac{\ln n}{\ln(n+2)} = 1$$

$$\Rightarrow \lim_{n\to\infty} \frac{(-1)^n \ln n}{\ln(n+2)} \text{ does not exist}$$

$\Rightarrow$ this series diverges.

3.b) $\sum_{n=1}^{\infty} \frac{2^{n^2}}{(n!)^n}$

Apply root test :

$$\lim_{n\to\infty} \sqrt[n]{\frac{2^{n^2}}{(n!)^n}} = \lim_{n\to\infty} \frac{2^n}{n!}$$

$$0 \leqslant \frac{2^n}{n!} = \left(\frac{2}{1}\right)\left(\frac{2}{2}\right)\left(\frac{2}{3}\right)\cdots\left(\frac{2}{n}\right) \leqslant \underbrace{\left(\frac{2}{1}\right)\left(\frac{2}{2}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\cdots\left(\frac{2}{3}\right)}_{2\left(\frac{2}{3}\right)^{n-2}}$$

$$\lim_{n\to\infty} 2\left(\frac{2}{3}\right)^{n-2} = \frac{9}{2} \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$$

because $0<\frac{2}{3}<1$

$$\lim_{n\to\infty} \frac{2^n}{n!} = 0 < 1 \Rightarrow \text{By root test this series converges.}$$

Problem 4 (10 points) Determine the real numbers p for which the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges.

Use the Integral Test:

$$\int \frac{dx}{x(\ln x)^p} = \int \frac{du}{u^p} = \begin{cases} \ln u & \text{for } p = 1 \\ \frac{u^{-p+1}}{1-p} & \text{for } p \neq 1 \end{cases}$$

$u := \ln x$

$\Downarrow$

$$\int_2^{\infty} \frac{dx}{x(\ln x)^p} = \begin{cases} \infty & \text{for } p \leq 1 \\ \frac{1}{p-1} & \text{for } p > 1 \end{cases}$$

$\Downarrow$

$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges for $p > 1$.

Problem 5.a (10 points) Obtain the Maclaurin series for the function $g(x) := \dfrac{1}{\sqrt{1-x^2}}$.

Hint: Use the definition of binomial series.

$$g(x) = (1-x^2)^{-1/2} = [1+(-x^2)]^{-1/2}$$

$$= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-n+1)}{n!} (-1)^n x^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{2n-1}{2})}{n!} (-1)^n x^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(1)(3)(5)\cdots(2n-1)}{2^n\, n!} x^{2n}$$

$$= 1 + \frac{x^2}{2} + \frac{3x^4}{8} + \cdots$$

Problem 5.b (5 points) Obtain the Maclaurin series for $\sin^{-1}(x)$.

Hint: Recall that $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + c$.

$$\sin^{-1}(x) = \int \frac{dx}{\sqrt{1-x^2}} = \int \left(\sum_{n=0}^{\infty} \frac{(1)(3)(5)\cdots(2n-1)}{2^n\, n!} x^{2n} \right) dx$$

$$= \sum_{n=0}^{\infty} \frac{(1)(3)(5)\cdots(2n-1)}{2^n\, n!} \left(\frac{x^{2n+1}}{2n+1} \right) + C$$

$$= x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots + C$$

$$\sin^{-1}(0) = 0 \Rightarrow C = 0$$

$$\Rightarrow \quad \sin^{-1}(x) = \sum_{n=0}^{\infty} \frac{(1)(3)(5)\cdots(2n-1)}{2^n\, n!\,(2n+1)} x^{2n+1}$$

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Hint: Use the definition of binomial series.

$$g(x) = (1-x^2)^{-1/2} = [1+(-x^2)]^{-1/2}$$

$$= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-n+1)}{n!} (-1)^n x^{2n}$$

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$$\sin^{-1}(0) = 0 \implies c = 0$$

$$\implies \sin^{-1}(x) = \sum_{n=0}^{\infty} \frac{(1)(3)(5)\cdots(2n-1)}{2^n\, n!\,(2n+1)} x^{2n+1}$$

Problem 5.c (5 points) Give the explicit expression for the the first three nonzero terms in the Maclaurin series for $\sin^{-1}(x)$ you find in Problem 5.b and use them together with the fact that $6\sin^{-1}(\frac{1}{2}) = \pi$, to obtain an approximate value for π.

$$\sin^{-1}(x) = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots$$

$$\pi = 6\sin^{-1}\left(\frac{1}{2}\right) \approx 6\left[\frac{1}{2} + \frac{1}{6}\left(\frac{1}{2}\right)^3 + \frac{3}{40}\left(\frac{1}{2}\right)^5\right]$$

$$\approx 3\left(1 + \frac{1}{24} + \frac{3}{20 \times 32}\right)$$

$$\approx 3 + \frac{1}{8} + \frac{9}{640}$$

$$\approx 3 + \frac{89}{640}$$

$$\frac{89}{640} \approx 0.138 \quad \Rightarrow \quad \pi \approx 3.138$$

Problem 6 (10 points) Find the sum of the power series $\sum_{n=0}^{\infty} n x^n$ for $|x| < 1$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for} \quad |x| < 1$$

$$\Rightarrow \frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{n=0}^{\infty} n x^{n-1}$$

$$\Rightarrow \sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}.$$

Problem 7 (15 points) Use the method of power series to obtain an approximate value for $\int_0^1 \ln(1+x^2)dx$ with an error less than 0.02.

$$\ln(1+x) = \int \frac{dx}{1+x} = \int \sum_{n=0}^{\infty} (-x)^n dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$$

$\ln(1) = 0 \Rightarrow C = 0$ for $|x| < 1$

$\Downarrow$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} \quad \text{for } |x| < 1$$

$\Uparrow$

$|x^2| < 1$

$\Downarrow$

$$\ln(1+x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{2n+2}$$

$\Downarrow$

$$\int_0^1 \ln(1+x^2)\, dx = \int_0^1 \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{2n+2} \right) dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(n+1)(2n+3)} \Bigg|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(2n+3)}$$

$$= \frac{1}{3} - \frac{1}{(2)(5)} + \frac{1}{(3)(7)} - \frac{1}{(4)(9)} + \frac{1}{(5)(11)} \pm \cdots$$

$$= \frac{1}{3} - \frac{1}{10} + \frac{1}{21} - \frac{1}{36} + \frac{1}{55}$$

Becau $\frac{1}{55} < 0.2$ we can keep the first 4 terms

So $\int_0^1 \ln(1+x^2)\, dx \approx \frac{1}{3} - \frac{1}{10} + \frac{1}{21} - \frac{1}{36} \approx \frac{1}{3}\underbrace{\left(1 + \frac{1}{7} - \frac{1}{12}\right)}_{} - \frac{1}{10}$

$$\approx \underbrace{\frac{89}{3 \times 84}}_{\approx 0.35} - \frac{1}{10} \approx 0.25 \pm 0.02$$

$$\frac{84 + 12 - 7}{84} = \frac{89}{84}$$