Chapter 2

Elements of Logic

2.1 Statements and Predicates

The building blocks of logical arguments are certain assertions called statements. *A **statement** is an assertion (sentence) that is either true or false.*[1] This means that *in order to establish that a statement is true it is sufficient to show that it is not false.* If a statement is true we say that its ***truth value*** is "True" (T). Similarly, a false statement has truth value "False" (F). The assertion that "1 is less than 2." is a an example of a statement with truth value "T." The assertion that "Every appel is red." is an example of a statement with truth value "F."

There are also assertions that are not statements, e.g., "x is an integer" which we may also express as "$x \in \mathbb{Z}$." This is an example of a predicate. *A **predicate** is an assertion involving one or more variables such that choosing a value for each of the variables turns the assertion into a statement.* For example, substituting 2 for x in the above predicate turns it into a true statement ($2 \in \mathbb{Z}$), whereas setting substituting $\frac{1}{2}$ for x turns it into a false statement ($\frac{1}{2} \in \mathbb{Z}$). An example of a predicate involving three variables is "$x \in \mathbb{N}, y \in \mathbb{Z}, \epsilon \in \mathbb{R}^+, x - y < \epsilon$." [2] We cannot decide if it is true unless we are provided with further information about the variables x, y and ϵ.

In studying logic we often deal with statements whose content is not specified. These are not to be confused with predicates. An unspecified statement $\mathfrak{a}$ is distinguished from other types of assertions by the conditions that it does not involve any variables and that it is either true or false; there is no other option for the truth value of a statement. The same holds for a predicate, but we are not able to

[1] Some books use the term "proposition" for what we call a "statement."
[2] Here we use the usual symbol "$<$" for "is less than."

determine the truth value of a predicate. We will also encounter predicates whose variable(s) are unspecified statements.

Definition 2.1.1 *Two statements* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are said to be* ***equal*** *if they have the same meaning, i.e., they can be used interchangeably in every argument. In this case we write* $\mathfrak{a} = \mathfrak{b}$. ■ [3]

This notion of equality of statements is not quite essential, because "$\mathfrak{a} = \mathfrak{b}$" simply means that $\mathfrak{a}$ is another symbol for $\mathfrak{b}$. Therefore we can completely avoid using "=" if we employ a unique notation for each statement appearing in a logical argument. In contrast, we always need to use the defining symbol ":=" (or "=:") whenever we introduce a new statement.

Consider the following logical argument concerning a statement $\mathfrak{a}$.

$\mathfrak{b}$:="If the statement that '$\mathfrak{a}$ is false' is false, then $\mathfrak{a}$ is true."

The reader undoubtedly agrees with the validity of this argument and that it is true independently of whether $\mathfrak{a}$ is itself true or false. For example, we may identify $\mathfrak{a}$ with the statement: "$1 > 2$." Then

$\mathfrak{b}$="If the statement that '$1 > 2$ is false' is false, then "$1 > 2$" is true."

Clearly, although $\mathfrak{a}$ is false, $\mathfrak{b}$ is true. As this example shows a valid logical argument, namely $\mathfrak{b}$ is always independent of the details of the circumstances it is applied to. Every logical argument is indeed a statement and what is important is its truth value. Indeed, we may describe Logic as a collection of rules that are used to deal with various statements without having a bearing on the details of their content but only their truth value. This is the main justification for the following definition.

Definition 2.1.2 *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be statements. Then* $\mathfrak{a}$ *is said to be* **logically equivalent** *to* $\mathfrak{b}$ *if* $\mathfrak{a}$ *and* $\mathfrak{b}$ *have the same truth value. In this case we write* $\mathfrak{a} \Leftrightarrow \mathfrak{b}$. ■

A trivial example of two logically equivalent statements is any statement $\mathfrak{a}$ and the statement "$\mathfrak{a}$ is true." We will encounter many nontrivial and useful examples of logically equivalent statements in the following sections.

The notions of equality and logical equivalence for statements may be extended to predicates.

Definition 2.1.3 *Two predicates are said to be* ***equal*** *if they have the same meaning, in particular they depend on the same variable(s). Two predicates are said to be* ***logically equivalent*** *if they depend on the same variable(s) and for each value of the variable(s) they yield logically equivalent statements.* ■

[3] We use ■ to mark the end of a definition, proof, or a solution.

2.2 Qualifiers

As we explained in the preceding section, fixing a particular value for the variable(s) of a predicate yields a statement. This is not the only way of turning a predicate into a statement. We can supplement the predicate with certain qualification of its variables. For example consider the predicate "x is greater than 1." We can turn this into a statement by qualifying x to be an arbitrary integer: "For all integer x, x is greater than 1." This is a false statement. Next consider qualifying x in a different way: "There is an integer x such that x is greater than 1." Clearly, this is a true statement. We will use the symbols "$\forall$" to mean "***for all***" and "$\exists$" for "***there exists one or many***." These are our basic ***qualifiers***. Using these symbols and the usual symbol "$>$" for "is greater than," we can express the preceding two statements as "$\forall x \in \mathbb{Z}$, $x > 1$" and "$\exists x \in \mathbb{Z}, x > 1$," respectively.

A simple property of ***qualified variables*** is that they can be freely relabeled; they ***are dummy variables***. For example, "$\forall x \in \mathbb{Z}$, $x > 1$" and "$\forall y \in \mathbb{Z}$, $y > 1$" are equal statements. Similarly, we have: $(\exists x \in \mathbb{Z},\ x > 1) = (\exists y \in \mathbb{Z},\ y > 1)$.

In qualifying the variables of a predicate, ***one must qualify each variable only once***. For example "$\exists x \in \mathbb{R}, \exists x \in \mathbb{Z}, x = 1$" is not an appropriate statement. Similarly, ***one must not use a single symbol for two different qualified variables***. For example, let $\mathfrak{a}_1 := (\exists x \in \mathbb{R}, 1 < x)$ and $\mathfrak{a}_2 := (\exists x \in \mathbb{R}, x < 0)$ which are true statements respectively obtained by qualifying the predicates $\mathfrak{p}_1(x) := (1 < x)$ and $\mathfrak{p}_2(x) := (x < 0)$. Now, because $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are true, there is a real number x such that $1 < x$ (according to $\mathfrak{a}_1$) and $x < 0$ (according to $\mathfrak{a}_2$). But, would not this imply $1 < 0$? The fallacy of this argument is in our illegitimate use of x for two different purposes, first as a qualified variable in $\mathfrak{a}_1$ and then as a qualified variable in $\mathfrak{a}_2$. To avoid this fallacy, we reserve x for the variable of $\mathfrak{p}_1$ that appears in $\mathfrak{a}_1$ and use y to denote the variable of $\mathfrak{p}_2$ that appears in $\mathfrak{a}_2$, i.e., write $\mathfrak{a}_2 := (\exists y \in \mathbb{R}, y < 0)$. In this way we are allowed to use x and y in other arguments, e.g., to establish the statement: "$\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, y < x$," that follows from $\mathfrak{a}_1$ and $\mathfrak{a}_2$ and the inequalities: $y < 0 < 1 < x$.

Next, consider comparing the statement $\mathfrak{a}$:= "for every real number x there is an integer n such that $x > n$," i.e., $\mathfrak{a} := (\forall x \in \mathbb{R}, \exists n \in \mathbb{Z},\ x > n)$, with the statement $\mathfrak{b} := (\exists n \in \mathbb{Z}, \forall x \in \mathbb{R},\ x > n)$. It is not difficult to see that $\mathfrak{a}$ is true while $\mathfrak{b}$ is false. Therefore, although $\mathfrak{b}$ is obtained from $\mathfrak{a}$ by changing the order in which "$\forall x \in \mathbb{R}$" and "$\exists n \in \mathbb{Z}$" appear, $\mathfrak{a}$ and $\mathfrak{b}$ are different statements. This example shows that ***changing the position of different terms appearing in a statement may change the statement altogether***.

In mathematical theories whenever one defines a new object, one must address the natural question of its *existence*. For example one may define "m"to be "the greatest natural number." But such a natural number does not exist. ***Defining***

a mathematical object does not imply its existence. The latter must be established independently.

Once one assures that a mathematical object exists, one must enquire into the question of its *uniqueness*. For example, let p and q be a pair of rational numbers satisfying $0 < p < q$ and $p^2 + q^2 = 1$. Such a pair exists because one can produce the example: $p = 3/5$ and $q = 4/5$. But this pair is not the only one. Another example is $p = 5/13$ and $q = 12/13$. This is a simple example of a mathematical problem whose solution exists but is not unique.

The existence and uniqueness problems are of fundamental importance in all areas of mathematics. The qualifier "$\exists!$" is often used to imply unique existence. It stands for "***there exists one and only one***." We can express the non-uniqueness discussed in the preceding paragraph as the statement:

$$\text{"}(\exists! p \in \mathbb{Q},\ \exists! q \in \mathbb{Q},\ 0 < p < q,\ p^2 + q^2 = 1) \text{ is false."}$$

The following is another example of a uniqueness statement.

$$\exists! m \in \mathbb{Z},\ \forall n \in \mathbb{Z},\ m + n = n.$$

2.3 Negation

To each statement we can associate another statement negating it.

Definition 2.3.1 *Let $\mathfrak{a}$ be a statement. Then the statement "$\mathfrak{a}$ is false."is called the **negation** of $\mathfrak{a}$ and denoted by $\neg\mathfrak{a}$.* ■

If $\mathfrak{a}$ happens to be true, then $\neg\mathfrak{a}$ is false and if $\mathfrak{a}$ is false then $\neg\mathfrak{a}$ is true. This shows that the truth value of $\neg\mathfrak{a}$ is the opposite of that of $\mathfrak{a}$. It is usually convenient to construct a table giving various possibilities for the truth values of unspecified statements. Such a table is called a ***truth table***. A simple example is Table 2.1. Its first column shows the two possible truth values of $\mathfrak{a}$. Its second column gives

$\mathfrak{a}$	$\neg\mathfrak{a}$
T	F
F	T

Table 2.1: Truth table for $\neg$

the corresponding truth values for $\neg\mathfrak{a}$. We may view Table 2.1 as an alternative definition of negation. We can use it to establish the following simple property of negation.

Proposition 2.3.1 *Let* $\mathfrak{a}$ *be a statement. Then* $\neg(\neg\mathfrak{a})$ *is logically equivalent to* $\mathfrak{a}$*, i.e.,* $(\neg(\neg\mathfrak{a})) \Leftrightarrow \mathfrak{a}$.

Proof: It suffices to extend Table 2.1 to include the truth values of $\neg(\neg\mathfrak{a})$. This yields Table 2.2 showing that $\mathfrak{a}$ and $\neg(\neg\mathfrak{a})$ have the same truth value. Hence, according to Definition 2.1.2, they are logically equivalent. ■

$\mathfrak{a}$	$\neg\mathfrak{a}$	$\neg(\neg\mathfrak{a})$
T	F	T
F	T	F

Table 2.2: Truth table for $\neg\neg$

We can extend the above definition of negation to predicates.

Definition 2.3.2 *Let* $\mathfrak{p}$ *be a predicate. Then the predicate "*$\mathfrak{p}$ *is false." is called the* ***negation*** *of* $\mathfrak{p}$ *and denoted by* $\neg\mathfrak{p}$. ■

Clearly, $\neg\mathfrak{p}$ depends on the same variable(s) as $\mathfrak{p}$ does. It is true (respectively false) for those values of the variable(s) for which $\mathfrak{p}$ is false (respectively true).

Next, we consider the problem of negating statements that involve qualifiers $\forall$, $\exists$, and $\exists!$.

Let $\mathfrak{a}$ be the statement: "$\forall n \in \mathbb{Z},\ n = 1$." To negate this statement, we must produce at least one integer that is different from 1. Expressing this in mathematical symbols we have: $\neg\mathfrak{a} = (\exists n \in \mathbb{Z},\ n \neq 1)$.[4] Next, consider the statement $\mathfrak{b} := (\exists r \in \mathbb{Q},\ r^2 + r^{-2} = 1)$. To negate $\mathfrak{b}$ we must show that for every rational number r the equality $r^2 + r^{-2} = 1$ is false. Therefore, $\neg\mathfrak{b} = (\forall r \in \mathbb{Q},\ r^2 + r^{-2} \neq 1)$. A straightforward application of this argument establishes the following theorem.

Theorem 2.3.1 *Let* $\mathfrak{p}(x)$ *be a predicate whose variables (collectively denoted by* x*) belong to a set* A*, and* $\mathfrak{c}$ *and* $\mathfrak{d}$ *be the statements:*

$$\mathfrak{c} := (\forall x \in A,\ \mathfrak{p}(x)), \qquad \mathfrak{d} := (\exists x \in A,\ \mathfrak{p}(x)). \tag{2.1}$$

Then

$$\neg\mathfrak{c} = (\exists x \in A,\ \neg\mathfrak{p}(x)), \qquad \neg\mathfrak{d} = (\forall x \in A,\ \neg\mathfrak{p}(x)). \tag{2.2}$$

Proof: Equations (2.2) follow from the same argument that we used to deal with the examples given in the preceding paragraph. ■

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[4] The reader should be able to justify our choice of using "=" in place of ":=" in the preceding relation and appreciate the fact that we could use another symbol say "m" in the expression for $\neg\mathfrak{a}$ instead of "n."

Exercise 2.3.1 Find $\neg\mathfrak{a}$ for $\mathfrak{a} := (\forall x \in \mathbb{R}, \exists n \in \mathbb{Z}, x < n)$.
Solution: First we express $\mathfrak{a}$ in one of the forms given in (2.1). We can do this by introducing $\mathfrak{p}(x) := (\exists n \in \mathbb{Z}, x < n)$, so that $\mathfrak{a} = (\forall x \in \mathbb{R}, \mathfrak{p}(x))$. Then in view of (2.2), we have

$$\neg\mathfrak{a} = (\exists x \in \mathbb{R}, \neg\mathfrak{p}(x)). \tag{2.3}$$

Now, we apply (2.2) once again for $\mathfrak{p}(x)$ to find $\neg\mathfrak{p}(x) = (\forall n \in \mathbb{Z}, x \geq n)$. Combining this relation with (2.3), we obtain

$$\neg\mathfrak{a} = (\exists x \in \mathbb{R}, \forall n \in \mathbb{Z}, x \geq n). \quad \blacksquare$$

This completes our discussion of negating statements involving $\forall$ and $\exists$. Next, we consider negating an statement involving $\exists!$.

Exercise 2.3.2 Find $\neg\mathfrak{u}$ for $\mathfrak{u} := (\exists! n \in \mathbb{Z}^+, n^2 < 2)$.
Solution: There are two ways in which we can negate $\mathfrak{u}$. Either we must show that there is no positive integer n satisfying $n^2 < 2$ or produce at least two (different) positive integers n_1 and n_2 such that $n_1^2 < 2$ and $n_2^2 < 2$. The first strategy indeed negates $\mathfrak{e} := (\exists n \in \mathbb{Z}^+, n^2 < 2)$ which amounts to $\neg\mathfrak{e} = (\forall n \in \mathbb{Z}^+, n^2 \geq 2)$. So let us assume that $\mathfrak{e}$ is true (as it is), and pursue the second strategy which is to actually negate the uniqueness feature of $\mathfrak{u}$. In mathematical symbols we can express it in the form:

$$\mathfrak{f} := (\exists n_1 \in \mathbb{Z}^+,\ \exists n_2 \in \mathbb{Z}^+,\ n_1 \neq n_2,\ n_1^2 < 2,\ n_2^2 < 2). \tag{2.4}$$

Strictly speaking, $\neg\mathfrak{u}$ asserts that either $\neg\mathfrak{e}$ or $\mathfrak{f}$ is true. This is an example of a compound statement. $\blacksquare$

2.4 Compound Statements

A typical mathematical argument involves a number of basic statements that are combined to form more complicated statements. These are called ***compound statements***. We have already encountered an example of a compound statement, namely $\mathfrak{a} \Leftrightarrow \mathfrak{b}$, that combines any two statements $\mathfrak{a}$ and $\mathfrak{b}$ into their logical equivalence. As we will see in this section, the logical equivalence may be expressed in terms of a pair of more elementary compound statements. There are three elementary compound statements.

Definition 2.4.1 *Let $\mathfrak{a}$ and $\mathfrak{b}$ be statements. Then*

(i) *"Both $\mathfrak{a}$ and $\mathfrak{b}$ are true." is called a **conjunction** and denoted by $\mathfrak{a} \wedge \mathfrak{b}$;*

(ii) *"Either* $\mathfrak{a}$ *or* $\mathfrak{b}$ *(or both) is true." is called a* ***disjunction*** *and denoted by* $\mathfrak{a} \vee \mathfrak{b}$*;*

(iii) *"If* $\mathfrak{a}$ *is true, then so is* $\mathfrak{b}$*." is called an* ***implication*** *and denoted by* $\mathfrak{a} \Rightarrow \mathfrak{b}$*.* ■

We may view $\wedge$, $\vee$, and $\Rightarrow$ as operations that apply to pairs of statements and yield the above compound statements. These together with the negation, $\neg$, provide the means for composing complicated statements out of the simpler ones. Indeed we use them whenever we conduct a logical argument. We have already encountered in Exercise 2.3.2 the need for a disjunction. We can express its solution as $\neg\mathfrak{u} = (\neg\mathfrak{e} \vee \mathfrak{f})$. We will encounter many examples of conjunctions, disjunctions, and implications in this book.

The truth value of a conjunction and a disjunction is evident from their definition. ***The only way in which*** $\mathfrak{a} \wedge \mathfrak{b}$ ***is true is that both*** $\mathfrak{a}$ ***and*** $\mathfrak{b}$ ***are true, and the only way in which*** $\mathfrak{a} \vee \mathfrak{b}$ ***is false is that both*** $\mathfrak{a}$ ***and*** $\mathfrak{b}$ ***are false.***

The constituent statements $\mathfrak{a}$ and $\mathfrak{b}$ of an implication $\mathfrak{a} \Rightarrow \mathfrak{b}$ are respectively called the ***hypothesis*** and the ***conclusion*** of the implication. To determine the truth value of an implication, it is convenient to ask when it is false. To falsify $\mathfrak{a} \Rightarrow \mathfrak{b}$, we must assure that its hypothesis $\mathfrak{a}$ is true but its conclusion $\mathfrak{b}$ is false. It is tempting to think of an implication as being inconclusive whenever its hypothesis is false. But being inconclusive is not an option for a statement. Because in the case that the hypothesis is false we cannot infer that the implication is false, it must be true. For example, let $\mathfrak{a}$:= "Mina has read this book.", $\mathfrak{b}$:= "Mina is able to solve all the exercise problems in this book.", and $\mathfrak{c} := (\mathfrak{a} \Rightarrow \mathfrak{b})$ = "If Mina has read this book, then she is able to solve all its exercises problems." The only way of refuting $\mathfrak{c}$ is to make sure that Mina has read the book but she cannot solve at least one of its exercise problems. Therefore, ***an implication*** $(\mathfrak{a} \Rightarrow \mathfrak{b})$ ***is true unless its hypothesis*** $(\mathfrak{a})$ ***is true and its conclusion*** $(\mathfrak{b})$ ***is false.***

Table 2.3 is the truth table for $\mathfrak{a} \wedge \mathfrak{b}$, $\mathfrak{a} \vee \mathfrak{b}$, and $\mathfrak{a} \Rightarrow \mathfrak{b}$. It has four rows, because there are a total of four possibilities for the truth values of $\mathfrak{a}$ and $\mathfrak{b}$.

$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{a} \wedge \mathfrak{b}$	$\mathfrak{a} \vee \mathfrak{b}$	$\mathfrak{a} \Rightarrow \mathfrak{b}$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

Table 2.3: Truth table for $\wedge$, $\vee$, and $\Rightarrow$

A basic property of conjunctions and disjunctions that follows from their definition is that up to logical equivalence the order in which we use them to combine two statements is irrelevant.

Theorem 2.4.1 *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be statements. Then*

$$(\mathfrak{a} \wedge \mathfrak{b}) \Leftrightarrow (\mathfrak{b} \wedge \mathfrak{a}), \qquad (\mathfrak{a} \vee \mathfrak{b}) \Leftrightarrow (\mathfrak{b} \vee \mathfrak{a}). \tag{2.5}$$

Proof: According to Table 2.3, the truth values of $\mathfrak{a} \wedge \mathfrak{b}$ and $\mathfrak{a} \vee \mathfrak{b}$ do not change if we swap $\mathfrak{a}$ and $\mathfrak{b}$. Therefore, $\mathfrak{a} \wedge \mathfrak{b}$ and $\mathfrak{b} \wedge \mathfrak{a}$ (respectively $\mathfrak{a} \vee \mathfrak{b}$ and $\mathfrak{b} \vee \mathfrak{a}$) have the same truth value, and in view of Definition 2.1.2 they are logically equivalent. ■

We will refer to this property of $\wedge$ and $\vee$ by saying that they are ***commutative*** operations.

Exercise 2.4.1 Construct the truth table for $\neg(\mathfrak{a} \wedge \mathfrak{b})$, $\neg(\mathfrak{a} \vee \mathfrak{b})$, and $\neg(\mathfrak{a} \Rightarrow \mathfrak{b})$.
Solution: The negation of a statement is true whenever the statement is false. Therefore, the truth table for $\neg(\mathfrak{a} \wedge \mathfrak{b})$, $\neg(\mathfrak{a} \vee \mathfrak{b})$, and $\neg(\mathfrak{a} \Rightarrow \mathfrak{b})$ is obtained by exchanging F and T in the last three columns of Table 2.3. The result is Table 2.4. ■

$\mathfrak{a}$	$\mathfrak{b}$	$\neg(\mathfrak{a} \wedge \mathfrak{b})$	$\neg(\mathfrak{a} \vee \mathfrak{b})$	$\neg(\mathfrak{a} \Rightarrow \mathfrak{b})$
T	T	F	F	F
T	F	T	F	T
F	T	T	F	F
F	F	T	T	F

Table 2.4: Truth table for the negation of $\wedge$, $\vee$, and $\Rightarrow$

Next, we derive some basic properties of logical equivalence.

Proposition 2.4.1 *Let* $\mathfrak{a}, \mathfrak{b}$ *and* $\mathfrak{c}$ *be statements. Then the following hold.*

(a) $(\mathfrak{a} = \mathfrak{b}) \Rightarrow (\mathfrak{a} \Leftrightarrow \mathfrak{b})$.

(b) $(\mathfrak{a} \Leftrightarrow \mathfrak{b}) \Leftrightarrow (\mathfrak{b} \Leftrightarrow \mathfrak{a})$.

(c) $((\mathfrak{a} \Leftrightarrow \mathfrak{b}) \wedge (\mathfrak{b} \Leftrightarrow \mathfrak{c})) \Rightarrow (\mathfrak{a} \Leftrightarrow \mathfrak{c}))$.

$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{a} \Leftrightarrow \mathfrak{b}$	$\mathfrak{b} \Leftrightarrow \mathfrak{a}$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

Table 2.5: Truth table establishing $(\mathfrak{a} \Leftrightarrow \mathfrak{b}) \Leftrightarrow (\mathfrak{b} \Leftrightarrow \mathfrak{a})$.

$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{a} \Leftrightarrow \mathfrak{b}$	$\mathfrak{b} \Leftrightarrow \mathfrak{c}$	$\mathfrak{a} \Leftrightarrow \mathfrak{c}$	$(\mathfrak{a} \Leftrightarrow \mathfrak{b}) \wedge (\mathfrak{b} \Leftrightarrow \mathfrak{c})$	$\mathfrak{d}$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	F	T	F	T
T	F	F	F	T	F	F	T
F	T	T	F	T	F	F	T
F	T	F	F	F	T	F	T
F	F	T	T	F	F	F	T
F	F	F	T	T	T	T	T

Table 2.6: Truth table showing that $\mathfrak{d} := (((\mathfrak{a} \Leftrightarrow \mathfrak{b}) \wedge (\mathfrak{b} \Leftrightarrow \mathfrak{c})) \Rightarrow (\mathfrak{a} \Leftrightarrow \mathfrak{c}))$ is true.

Proof: According to the definition of logical equivalence (Definition 2.1.2), (a) holds because equal statements have equal truth values. To establish (b) and (c) we construct the relevant truth tables namely Tables 2.5 and 2.6. According to Table 2.5, $\mathfrak{a} \Leftrightarrow \mathfrak{b}$ and $\mathfrak{b} \Leftrightarrow \mathfrak{a}$ have the same truth value. This proves (b). Similarly Table (2.6) shows that (c) is also true. ■

Part (b) of this proposition indicates that $\Leftrightarrow$ is a commutative operation. This justifies the identification of the statements "$\mathfrak{a}$ is logically equivalent to $\mathfrak{b}$" and "$\mathfrak{a}$ and $\mathfrak{b}$ are logically equivalent."

Proposition 2.4.2 *Let $\mathfrak{a}$ and $\mathfrak{b}$ be statements. If $\mathfrak{a}$ is logically equivalent to $\mathfrak{b}$, then $\neg\mathfrak{a}$ is logically equivalent to $\neg\mathfrak{b}$, i.e.,* $(\mathfrak{a} \Leftrightarrow \mathfrak{b}) \Rightarrow (\neg\mathfrak{a} \Leftrightarrow \neg\mathfrak{b})$.[5]
Proof: If $\mathfrak{a} \Leftrightarrow \mathfrak{b}$, then $\mathfrak{a}$ and $\mathfrak{b}$ have the same truth value. But then $\neg\mathfrak{a}$ has the

[5] **Convention:** In compound statements involving $\neg$, this symbol is assumed to affect only the first statement to its right. Parenthesis are used to negate compound statements, e.g., $\neg(\mathfrak{a} \Rightarrow \neg\mathfrak{b})$. We have adopted this convention to reduce the number of parenthesis appearing in more complicated compound statements. For example it allow to write $(\neg\mathfrak{a}) \wedge (\neg\mathfrak{b})$ as $\neg\mathfrak{a} \wedge \neg\mathfrak{b}$.

opposite truth value to $\mathfrak{b}$. Hence $\neg\mathfrak{a}$ has the same truth value as $\neg\mathfrak{b}$. In view of Definition 2.1.2, this implies $\neg\mathfrak{a} \Leftrightarrow \neg\mathfrak{b}$. ■

Next, we show that unlike $\wedge$, $\vee$, and $\Leftrightarrow$, the operation $\Rightarrow$ is not commutative. ***In general, changing the roles of the hypothesis and the conclusion of an implication changes the implication itself.*** This is actually quite evident. For example, consider the implication: "If Mina has read this book, then she can solve all its exercise problems." Changing the hypothesis and the conclusion of this implication yields: "If Mina can solve all the exercise problems in this book, then she has read it." These two statements are clearly different. It is quite possible that the first is true while the second is not.

Proposition 2.4.3 *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be arbitrary statements. Then* $\mathfrak{a} \Rightarrow \mathfrak{b}$ *and* $\mathfrak{b} \Rightarrow \mathfrak{a}$ *are not logically equivalent.*
Proof: It is sufficient to show that $\mathfrak{a} \Rightarrow \mathfrak{b}$ and $\mathfrak{b} \Rightarrow \mathfrak{a}$ do not have the same truth value. But this is evident from Table 2.7. Comparing the third and forth columns of this table, we see that indeed $\mathfrak{a} \Rightarrow \mathfrak{b}$ and $\mathfrak{b} \Rightarrow \mathfrak{a}$ have different truth values. ■

As it is clear from Table 2.3, the truth value of an implication must not be confused with the truth value of its conclusion. It is possible for an implication to be true even if its conclusion is false. This happens precisely in the case that the hypothesis is also false. In the above-mentioned example, we recall that if Mina has not read the book and she cannot solve all the exercise problems, we cannot say that the implication "If Mina has read this book, then she can solve all its exercise problems" is false. Hence it is true.

If the implication $\mathfrak{a} \Rightarrow \mathfrak{b}$ is true, one says that $\mathfrak{a}$ is a ***sufficient condition*** for $\mathfrak{b}$ and $\mathfrak{b}$ is a ***necessary condition*** for $\mathfrak{a}$. It is also customary to use the phrase "$\mathfrak{a}$ ***if and only if*** $\mathfrak{b}$" or its abbreviation: "$\mathfrak{a}$ ***iff*** $\mathfrak{b}$" for the logical equivalence $\mathfrak{a} \Leftrightarrow \mathfrak{b}$. In this case one says that $\mathfrak{a}$ is a ***necessary and sufficient condition*** for $\mathfrak{b}$. The following characterization of logical equivalence justifies this terminology.

Theorem 2.4.2 *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be statements. Then* $\mathfrak{a} \Leftrightarrow \mathfrak{b}$ *is logically equivalent to* $(\mathfrak{a} \Rightarrow \mathfrak{b}) \wedge (\mathfrak{b} \Rightarrow \mathfrak{a})$.
Proof: As shown in Table 2.7, $\mathfrak{a} \Leftrightarrow \mathfrak{b}$ has the same truth value as $(\mathfrak{a} \Rightarrow \mathfrak{b}) \wedge (\mathfrak{b} \Rightarrow \mathfrak{a})$. Hence they are logically equivalent. ■

This theorem shows that in order ***to prove the logical equivalence of two statements*** $\mathfrak{a}$ ***and*** $\mathfrak{b}$***, one must establish both the implication*** $\mathfrak{a} \Rightarrow \mathfrak{b}$ ***and its converse*** $\mathfrak{b} \Rightarrow \mathfrak{a}$***.***

Corollary 2.4.1 *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be statements. Then* $\mathfrak{a} \Leftrightarrow \mathfrak{b}$ *is logically equivalent to* $\neg\mathfrak{a} \Leftrightarrow \neg\mathfrak{b}$*, i.e.,* $(\mathfrak{a} \Leftrightarrow \mathfrak{b}) \Leftrightarrow (\neg\mathfrak{a} \Leftrightarrow \neg\mathfrak{b})$.

$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{a} \Leftrightarrow \mathfrak{b}$	$\mathfrak{a} \Rightarrow \mathfrak{b}$	$\mathfrak{b} \Rightarrow \mathfrak{a}$	$(\mathfrak{a} \Rightarrow \mathfrak{b}) \wedge (\mathfrak{b} \Rightarrow \mathfrak{a})$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Table 2.7: Truth table establishing $(\mathfrak{a} \Leftrightarrow \mathfrak{b}) \Leftrightarrow ((\mathfrak{a} \Rightarrow \mathfrak{b}) \wedge (\mathfrak{b} \Rightarrow \mathfrak{a}))$.

Proof: To prove this statement we prove $\mathfrak{c} := ((\mathfrak{a} \Leftrightarrow \mathfrak{b}) \Rightarrow (\neg\mathfrak{a} \Leftrightarrow \neg\mathfrak{b}))$ and $\mathfrak{d} := ((\neg\mathfrak{a} \Leftrightarrow \neg\mathfrak{b}) \Rightarrow (\mathfrak{a} \Leftrightarrow \mathfrak{b}))$. We have shown $\mathfrak{c}$ in Proposition 2.4.2. To show $\mathfrak{d}$ we apply $\mathfrak{c}$ for the case that $\mathfrak{a}$ and $\mathfrak{b}$ are respectively replaced with their negations. This together with Proposition 2.3.1 yield $\mathfrak{d}$. ■

Logical equivalence plays a fundamental role in logical arguments. If we replace the constituent statements $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \cdots$ of a compound statement $\mathfrak{b}$ by the statements $\mathfrak{a}'_1, \mathfrak{a}'_2, \mathfrak{a}'_3, \cdots$ that are respectively logically equivalent to $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \cdots$, we obtain a new compound statement $\mathfrak{b}'$ which is logically equivalent to $\mathfrak{b}$. In other words, we can use logically equivalent statements interchangeably in logical arguments. For example in order to establish the validity of an implication $\mathfrak{a}_1 \Rightarrow \mathfrak{a}_2$ we may as well find statements $\mathfrak{a}'_1$ and $\mathfrak{a}'_2$ which are respectively logically equivalent to $\mathfrak{a}_1$ and $\mathfrak{a}_2$ and establish the implication $\mathfrak{a}'_1 \Rightarrow \mathfrak{a}'_2$. We leave the proof of this statement for the reader (Problem 2.6).

Often the definition of a mathematical object does not provide a useful method of identifying concrete examples of such an object. This motivates finding alternative conditions that are both necessary and sufficient for the validity of the defining conditions of the object in question. The statement establishing the logical equivalence of these two sets of conditions is called a ***characterization theorem***. A typical example is Theorem 2.4.2 that provides a necessary and sufficient condition for the logical equivalence of two statements. Often characterization theorems assert the logical equivalence of several statements say $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_n$ for some $n \in \mathbb{Z}^+$. To prove such a theorem one must establish the logical equivalence of $\mathfrak{a}_i$ and $\mathfrak{a}_j$ for all i and j between 1 and n. Given that $\Leftrightarrow$ is commutative and every statement is logically equivalent to itself, this amounts to proving $\frac{1}{2}n(n-1)$ logical equivalences or, in view of Theorem 2.4.2, $n(n-1)$ implications. It turns out, however, that it is sufficient (and necessary) to prove the following complete cycle of n implications.

$$(\mathfrak{a}_1 \Rightarrow \mathfrak{a}_2) \wedge (\mathfrak{a}_2 \Rightarrow \mathfrak{a}_3) \wedge \cdots \wedge (\mathfrak{a}_{n-1} \Rightarrow \mathfrak{a}_n) \wedge (\mathfrak{a}_n \Rightarrow \mathfrak{a}_1). \tag{2.6}$$

Here we prove this statement for $n = 3$.

Theorem 2.4.3 *Let* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ *be statements and*

$$\mathfrak{d} := ((\mathfrak{a} \Leftrightarrow \mathfrak{b}) \wedge (\mathfrak{a} \Leftrightarrow \mathfrak{c})) \wedge (\mathfrak{b} \Leftrightarrow \mathfrak{c})), \qquad \mathfrak{e} := ((\mathfrak{a} \Rightarrow \mathfrak{b}) \wedge (\mathfrak{b} \Rightarrow \mathfrak{c}) \wedge (\mathfrak{c} \Rightarrow \mathfrak{a}))$$

Then $\mathfrak{d}$ *and* $\mathfrak{e}$ *are logically equivalent.*
Proof: We use Tables 2.3 and 2.7 to construct the relevant truth table for the problem (Table 2.8) and check that $\mathfrak{d}$ and $\mathfrak{e}$ have the same truth value. ■

$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{a} \Rightarrow \mathfrak{b}$	$\mathfrak{b} \Rightarrow \mathfrak{c}$	$\mathfrak{c} \Rightarrow \mathfrak{a}$	$\mathfrak{a} \Leftrightarrow \mathfrak{b}$	$\mathfrak{a} \Leftrightarrow \mathfrak{c}$	$\mathfrak{b} \Leftrightarrow \mathfrak{c}$	$\mathfrak{d}$	$\mathfrak{e}$
T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	F	T	T	F	F	F	F
T	F	T	F	T	T	F	T	F	F	F
T	F	F	F	T	T	F	F	T	F	F
F	T	T	T	T	F	F	F	T	F	F
F	T	F	T	F	T	F	T	F	F	F
F	F	T	T	T	F	T	F	F	F	F
F	F	F	T	T	T	T	T	T	T	T

Table 2.8: Truth table establishing the logical equivalence of $\mathfrak{d} := ((\mathfrak{a} \Leftrightarrow \mathfrak{b}) \wedge (\mathfrak{a} \Leftrightarrow \mathfrak{c})) \wedge (\mathfrak{b} \Leftrightarrow \mathfrak{c}))$ and $\mathfrak{e} := ((\mathfrak{a} \Rightarrow \mathfrak{b}) \wedge (\mathfrak{b} \Rightarrow \mathfrak{c}) \wedge (\mathfrak{c} \Rightarrow \mathfrak{a}))$.

2.5 Contradictions and Tautologies

Consider the following two compound statements that depend on an unspecified statement $\mathfrak{b}$.

$$\mathfrak{a} := (\mathfrak{b} \wedge \neg\mathfrak{b}), \qquad \mathfrak{c} := (\mathfrak{b} \vee \neg\mathfrak{b}).$$

Clearly, $\mathfrak{a}$ is false and $\mathfrak{c}$ is true regardless of whether $\mathfrak{b}$ is true or false. $\mathfrak{a}$ and $\mathfrak{b}$ are examples of a contradiction and a tautology, respectively.

Definition 2.5.1 *A* ***contradiction*** *is a compound statement which is false regardless of the truth value of its constituents statements. Similarly, a* ***tautology*** *is a compound statement which is true regardless of the truth value of its constituents statements.* ■

Exercise 2.5.1 Let $\mathfrak{a}$ be a statement and $\mathfrak{b} := (\mathfrak{a} \Rightarrow \neg\mathfrak{a})$. Is $\mathfrak{b}$ a contradiction?
Solution: $\mathfrak{b}$ is an implication. If its hypothesis ($\mathfrak{a}$) is false, $\mathfrak{b}$ is true. This shows that $\mathfrak{b}$ is not a contradiction. ■

A simple consequence of Definition 2.5.1 is the following uniqueness theorem.

Theorem 2.5.1 *Let $\mathfrak{c}$ and $\mathfrak{c}'$ be contradictions and $\mathfrak{t}$ and $\mathfrak{t}'$ be tautologies. Then*

$$\mathfrak{c} \Leftrightarrow \mathfrak{c}', \qquad \mathfrak{t} \Leftrightarrow \mathfrak{t}', \qquad \mathfrak{c} \Leftrightarrow \neg\mathfrak{t}. \tag{2.7}$$

Proof: These follow directly from Definition 2.5.1 and Theorem 2.4.2. Because $\mathfrak{c}$ and $\mathfrak{c}'$ are both false, they have the same truth value. So they are logically equivalent. The same argument applies to $\mathfrak{t}$ and $\mathfrak{t}'$, because both of them have truth value "T." Finally $\neg\mathfrak{t}$ has truth value "F" so it is a contradiction. As a result, we may identify $\neg\mathfrak{t}$ with $\mathfrak{c}'$ in the first relation in (2.7). This yields $\mathfrak{c} \Leftrightarrow \neg\mathfrak{t}$. ■

This theorem indicates that up to logical equivalence there are a unique contradiction and a unique tautology, and that the former is the negation of the latter.

Exercise 2.5.2 Let $\mathfrak{a}, \mathfrak{c}$, and $\mathfrak{t}$ be respectively an arbitrary statement, a contradiction, and a tautology. Show that $\mathfrak{a} \Rightarrow \mathfrak{t}$ and $\mathfrak{c} \Rightarrow \mathfrak{a}$ are tautologies.
Solution: Because $\mathfrak{t}$ is true, according to Table 2.3, $\mathfrak{a} \Rightarrow \mathfrak{t}$ is true irrespective of whether $\mathfrak{a}$ is true or false. Therefore, $\mathfrak{a} \Rightarrow \mathfrak{t}$ is a tautology. Similarly, $\mathfrak{c} \Rightarrow \mathfrak{a}$ is a tautology, because $\mathfrak{c}$ is false and according to Table 2.3 this suffices to hold that $\mathfrak{c} \Rightarrow \mathfrak{a}$ is true regardless of the truth value of $\mathfrak{a}$. ■

A strange outcome of this exercise is that contradictions imply tautologies! The reader must not view all tautologies as unimportant or useless. For example, consider the statement (c) of Proposition 2.4.1, i.e.,

$$\mathfrak{d} := (((\mathfrak{a} \Leftrightarrow \mathfrak{b}) \wedge (\mathfrak{b} \Leftrightarrow \mathfrak{c})) \Rightarrow (\mathfrak{a} \Leftrightarrow \mathfrak{c})).$$

Since we have proven that $\mathfrak{d}$ is true regardless of the nature of its constituent statements, $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$, by Definition 2.5.1, $\mathfrak{d}$ is a tautology! Indeed, a large number of theorems in mathematics concern establishing that certain compound statements are tautologies. The following theorem is an example. It provides the basis for one of the most important methods of establishing the validity of an implication, namely the method of *proof by deduction* (Section 3.4).

Theorem 2.5.2 (Two-step deduction) *Let $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ be statements. Then the statement $\mathfrak{d} := ((\mathfrak{a} \Rightarrow \mathfrak{c}) \wedge (\mathfrak{c} \Rightarrow \mathfrak{b}))$ implies $\mathfrak{a} \Rightarrow \mathfrak{b}$, i.e., $\mathfrak{e} := (\mathfrak{d} \Rightarrow (\mathfrak{a} \Rightarrow \mathfrak{b}))$ is a tautology.*
Proof: We determine the truth value of $\mathfrak{e}$ by considering all possible truth values of $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$. Constructing the relevant truth table (Table 2.9) we find that indeed $\mathfrak{e}$ is always true; it is a tautology. ■

Exercise 2.5.3 *Let $\mathfrak{e}, \mathfrak{f}$ and $\mathfrak{g}$ be statements. Show that*

$$\mathfrak{h} := (\,((\mathfrak{e} \wedge \mathfrak{f}) \wedge (\mathfrak{e} \Rightarrow \mathfrak{g})) \;\Rightarrow\; (\mathfrak{f} \wedge \mathfrak{g})\,)$$

$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{a} \Rightarrow \mathfrak{c}$	$\mathfrak{c} \Rightarrow \mathfrak{b}$	$\mathfrak{a} \Rightarrow \mathfrak{b}$	$\mathfrak{d}$	$\mathfrak{e}$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	F	T
T	F	T	T	F	F	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	F	T	F	T
F	F	F	T	T	T	T	T

Table 2.9: Truth table establishing that $\mathfrak{d} := ((\mathfrak{a} \Rightarrow \mathfrak{c}) \wedge (\mathfrak{c} \Rightarrow \mathfrak{b}))$ implies $\mathfrak{a} \Rightarrow \mathfrak{b}$. Here $\mathfrak{e} := (\mathfrak{d} \Rightarrow (\mathfrak{a} \Rightarrow \mathfrak{b}))$.

is a tautology.
Solution: Again we can establish $\mathfrak{h}$ by constructing its truth table (Problem 2.3). Here we give an alternative proof that is based on our knowledge of implications and conjunctions. Our aim is to show that $\mathfrak{h}$ cannot be false. First, we recall that an implication is false only if its hypothesis is true and its conclusion is false, and a conjunction is true only if its constituent statements are both true. We start our argument by expressing $\mathfrak{h}$ as the implication: $\mathfrak{h} = (\mathfrak{a} \Rightarrow \mathfrak{b})$ where

$$\begin{aligned} \mathfrak{a} &:= ((\mathfrak{e} \wedge \mathfrak{f}) \wedge (\mathfrak{e} \Rightarrow \mathfrak{g})), \\ \mathfrak{b} &:= (\mathfrak{f} \wedge \mathfrak{g}). \end{aligned}$$

$\mathfrak{h}$ can be false only if $\mathfrak{a}$ is true and $\mathfrak{b}$ is false. To ensure that $\mathfrak{a}$ is true,

(1) $\mathfrak{e} \wedge \mathfrak{f}$ must be true, which implies $\mathfrak{e}$ and $\mathfrak{f}$ are both true, and

(2) $\mathfrak{e} \Rightarrow \mathfrak{g}$ must be true.

Combining (1) and (2), we see that because both $\mathfrak{e}$ and $\mathfrak{e} \Rightarrow \mathfrak{g}$ are true, $\mathfrak{g}$ must be true. But according to (1), $\mathfrak{f}$ is also true. This shows that there is no way we can ensure that $\mathfrak{b}$ is false. Therefore, it is impossible for $\mathfrak{h}$ to be false; it is true regardless of the truth values of its constituents, i.e., it is a tautology. ■

Our solution of Exercise 2.5.3 involves two parts. First we actually consider the possibility that the statement we wish to prove is false. We then show that this never happens. This approach is called the method of *proof by contradiction* that we will examine more thoroughly in Section 3.5. We use a similar approach to solve the following exercise problem.

Exercise 2.5.4 *Let* $\mathfrak{e}, \mathfrak{f}$ *and* $\mathfrak{g}$ *be statements. Show that*

$$\mathfrak{h} := (\ (\mathfrak{e} \Rightarrow \mathfrak{f})\ \Rightarrow\ ((\mathfrak{e} \wedge \mathfrak{g}) \Rightarrow (\mathfrak{f} \wedge \mathfrak{g}))\ )$$

is a tautology.
Solution: We can view $\mathfrak{h}$ as an implication with $\mathfrak{a} := (\mathfrak{e} \Rightarrow \mathfrak{f})$ as its hypothesis and $\mathfrak{b} := ((\mathfrak{e} \wedge \mathfrak{g}) \Rightarrow (\mathfrak{f} \wedge \mathfrak{g}))$ as its conclusion. The only way in which $\mathfrak{h}$ can be false is that $\mathfrak{a}$ is true and $\mathfrak{b}$ is false. The latter requires $\mathfrak{e} \wedge \mathfrak{g}$ to be true and $\mathfrak{f} \wedge \mathfrak{g}$ to be false. The first of these implies that both $\mathfrak{e}$ and $\mathfrak{g}$ must be true, but then the second can be achieved only if $\mathfrak{f}$ is false. Next, we consider $\mathfrak{a}$ which we also view as an implication. Because its conclusion ($\mathfrak{f}$) is false and $\mathfrak{a}$ is true, its hypothesis $\mathfrak{e}$ must be false as well. But we already established that $\mathfrak{e}$ is true. This argument shows that $\mathfrak{h}$ cannot be false. Therefore, it is always true regardless of the truth values of $\mathfrak{e}, \mathfrak{f}$ and $\mathfrak{g}$; it is a tautology. ■

2.6 Propositional Calculus

The compound statements we have so far introduced are actually not completely independent. We have already related logical equivalence to a conjunction of a pair of implications (Theorem 2.4.2). In this section we reveal some basic relationships between the elementary compound statements. We begin linking conjunctions and disjunctions.

Theorem 2.6.1 (De Morgan's laws) *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be statements, then*

(a) $\neg(\mathfrak{a} \wedge \mathfrak{b})$ *and* $\neg\mathfrak{a} \vee \neg\mathfrak{b}$ *are logically equivalent;*

(b) $\neg(\mathfrak{a} \vee \mathfrak{b})$ *and* $\neg\mathfrak{a} \wedge \neg\mathfrak{b}$ *are logically equivalent.*

Proof: To establish (a), we construct the relevant truth table (Table 2.10), and realize that $\neg(\mathfrak{a} \wedge \mathfrak{b})$ and $\neg\mathfrak{a} \vee \neg\mathfrak{b}$ have identical truth values. To prove (b) we can

$\mathfrak{a}$	$\mathfrak{b}$	$\neg\mathfrak{a}$	$\neg\mathfrak{b}$	$\mathfrak{a} \wedge \mathfrak{b}$	$\neg(\mathfrak{a} \wedge \mathfrak{b})$	$\neg\mathfrak{a} \vee \neg\mathfrak{b}$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Table 2.10: Truth table establishing $(\neg(\mathfrak{a} \wedge \mathfrak{b})) \Leftrightarrow (\neg\mathfrak{b} \vee \neg\mathfrak{a})$

construct the corresponding truth table (Problem 2.3) or reduce its proof to that of

(a) as we explain next. In view of Proposition 2.3.1, which asserts $(\neg(\neg\mathfrak{e})) \Leftrightarrow \mathfrak{e}$ for any statement $\mathfrak{e}$, and part (a) above, we have for any pair of statements $\mathfrak{c}$ and $\mathfrak{d}$,

$$(\mathfrak{c} \wedge \mathfrak{d}) \Leftrightarrow (\neg(\neg\mathfrak{c} \vee \neg\mathfrak{d})). \tag{2.8}$$

Now, let $\mathfrak{c} := \neg\mathfrak{a}$ and $\mathfrak{d} := \neg\mathfrak{b}$, so that $\mathfrak{a} \Leftrightarrow \neg\mathfrak{c}$ and $\mathfrak{b} \Leftrightarrow \neg\mathfrak{d}$. Substituting these relations in (2.8), we have

$$(\neg\mathfrak{a} \wedge \neg\mathfrak{b}) \Leftrightarrow (\neg(\mathfrak{a} \vee \mathfrak{b})),$$

which in view of the commutativity of $\Leftrightarrow$ (Theorem 2.4.1) establishes (b). ■

A simple consequence of the preceding theorem is that the operations $\neg, \wedge$, and $\vee$ are not independent; ***up to logical equivalence we can use*** $\neg$ ***and*** $\wedge$ ***to express*** $\vee$ ***and use*** $\neg$ ***and*** $\vee$ ***to express*** $\wedge$:

$$(\mathfrak{a} \vee \mathfrak{b}) \Leftrightarrow (\neg(\neg\mathfrak{a} \wedge \neg\mathfrak{b})), \qquad (\mathfrak{a} \wedge \mathfrak{b}) \Leftrightarrow (\neg(\neg\mathfrak{a} \vee \neg\mathfrak{b})). \tag{2.9}$$

Next, we relate $\Rightarrow$ to $\vee$ and consequently $\wedge$.

Theorem 2.6.2 *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be statements. Then* $\mathfrak{a} \Rightarrow \mathfrak{b}$ *is logically equivalent to* $\neg\mathfrak{a} \wedge \mathfrak{b}$, *i.e.,*

$$(\mathfrak{a} \Rightarrow \mathfrak{b}) \Leftrightarrow (\neg\mathfrak{a} \vee \mathfrak{b}). \tag{2.10}$$

Proof: This follows from Table 2.11. ■

$\mathfrak{a}$	$\mathfrak{b}$	$\neg\mathfrak{a}$	$\neg\mathfrak{a} \vee \mathfrak{b}$	$\mathfrak{a} \Rightarrow \mathfrak{b}$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Table 2.11: Truth table establishing $(\mathfrak{a} \Rightarrow \mathfrak{b}) \Leftrightarrow (\neg\mathfrak{a} \vee \mathfrak{b})$

Notation: Let $\mathfrak{c}_1, \mathfrak{c}_2, \cdots, \mathfrak{c}_n$ be statements. Then

$$\mathfrak{c}_1 \wedge \mathfrak{c}_2 \wedge \cdots \wedge \mathfrak{c}_n \;:=\; \mathfrak{c}_1 \wedge (\mathfrak{c}_2 \wedge (\mathfrak{c}_3 \wedge \cdots (\mathfrak{c}_{n-1} \wedge \mathfrak{c}_n) \cdots)), \tag{2.11}$$

$$\mathfrak{c}_1 \vee \mathfrak{c}_2 \vee \cdots \vee \mathfrak{c}_n \;:=\; \mathfrak{c}_1 \vee (\mathfrak{c}_2 \vee (\mathfrak{c}_3 \vee \cdots (\mathfrak{c}_{n-1} \vee \mathfrak{c}_n) \cdots)), \tag{2.12}$$

$$\mathfrak{c}_1 \Rightarrow \mathfrak{c}_2 \Rightarrow \cdots \Rightarrow \mathfrak{c}_n \;:=\; ((\mathfrak{c}_1 \Rightarrow \mathfrak{c}_2) \wedge (\mathfrak{c}_2 \Rightarrow \mathfrak{c}_3) \wedge \cdots \wedge (\mathfrak{c}_{n-1} \Rightarrow \mathfrak{c}_n)), \tag{2.13}$$

$$\mathfrak{c}_1 \Leftrightarrow \mathfrak{c}_2 \Leftrightarrow \cdots \Leftrightarrow \mathfrak{c}_n \;:=\; ((\mathfrak{c}_1 \Leftrightarrow \mathfrak{c}_2) \wedge (\mathfrak{c}_2 \Leftrightarrow \mathfrak{c}_3) \wedge \cdots \wedge (\mathfrak{c}_{n-1} \Leftrightarrow \mathfrak{c}_n)). \tag{2.14}$$

Corollary 2.6.1 *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be statements. Then* $\mathfrak{a} \Rightarrow \mathfrak{b}$ *is logically equivalent to* $\neg\mathfrak{b} \Rightarrow \neg\mathfrak{a}$*, i.e.,*

$$(\mathfrak{a} \Rightarrow \mathfrak{b}) \Leftrightarrow (\neg\mathfrak{b} \Rightarrow \neg\mathfrak{a}). \tag{2.15}$$

Proof: Applying (2.10) to $\neg\mathfrak{b} \Rightarrow \neg\mathfrak{a}$, using the commutativity of $\vee$, and then applying (2.10) again, we have

$$(\neg\mathfrak{b} \Rightarrow \neg\mathfrak{a}) \Leftrightarrow (\neg(\neg\mathfrak{b}) \vee \neg\mathfrak{a}) \Leftrightarrow (\mathfrak{b} \vee \neg\mathfrak{a}) \Leftrightarrow (\neg\mathfrak{a} \vee \mathfrak{b}) \Leftrightarrow (\mathfrak{a} \Rightarrow \mathfrak{b}). \tag{2.16}$$

This relation together with part (b) of Proposition 2.4.1 and the commutativity of $\Leftrightarrow$ yield (2.15). ■

The right-hand side of the logical equivalence (2.15) is called the ***contrapositive*** of its left-hand side. In Section 3.4, we will use this logical equivalence to outline a method of proving implications which is called the *contrapositive proof.* It relies on the simple observation that in order to prove an implication it is sufficient to prove its contrapositive.

Theorem 2.6.3 *Let* $\mathfrak{a}, \mathfrak{b}$ *and* $\mathfrak{c}$ *be statements. Then*

$$(\mathfrak{a} \wedge (\mathfrak{b} \wedge \mathfrak{c})) \Leftrightarrow ((\mathfrak{a} \wedge \mathfrak{b}) \wedge \mathfrak{c}), \tag{2.17}$$
$$(\mathfrak{a} \vee (\mathfrak{b} \vee \mathfrak{c})) \Leftrightarrow ((\mathfrak{a} \vee \mathfrak{b}) \vee \mathfrak{c}), \tag{2.18}$$
$$(\mathfrak{a} \wedge (\mathfrak{b} \vee \mathfrak{c})) \Leftrightarrow ((\mathfrak{a} \wedge \mathfrak{b}) \vee (\mathfrak{a} \wedge \mathfrak{c})), \tag{2.19}$$
$$(\mathfrak{a} \vee (\mathfrak{b} \wedge \mathfrak{c})) \Leftrightarrow ((\mathfrak{a} \vee \mathfrak{b}) \wedge (\mathfrak{a} \vee \mathfrak{c})). \tag{2.20}$$

Proof: The above logical equivalences can be established most easily by constructing the corresponding truth tables. We do this for (2.19) leaving the others for the reader. The truth table proving (2.19) is Table 2.12. It shows that $\mathfrak{a} \wedge (\mathfrak{b} \vee \mathfrak{c})$ and $(\mathfrak{a} \wedge \mathfrak{b}) \vee (\mathfrak{a} \wedge \mathfrak{c})$ have the same truth value. Hence they are logically equivalent. ■

The logical equivalences (2.17) and (2.18) are usually referred to as the ***associativity*** of the operations $\wedge$ and $\vee$. Similarly (2.19) and (2.20) are the statements of the ***distribution laws*** of $\wedge$ over $\vee$ and $\vee$ over $\wedge$, respectively. Note that because $\wedge$ and $\vee$ are commutative operations, the following distribution laws hold as well.

$$((\mathfrak{a} \vee \mathfrak{b}) \wedge \mathfrak{c}) \Leftrightarrow ((\mathfrak{a} \wedge \mathfrak{c}) \vee (\mathfrak{b} \wedge \mathfrak{c})), \tag{2.21}$$
$$((\mathfrak{a} \wedge \mathfrak{b}) \vee \mathfrak{c}) \Leftrightarrow ((\mathfrak{a} \vee \mathfrak{c}) \wedge (\mathfrak{b} \vee \mathfrak{c})). \tag{2.22}$$

It is not difficult to show that $\Leftrightarrow$ is also an associative operation (Problem 2.4).

The operations $\vee$ and $\wedge$ share many (but not all) the properties of addition and multiplication of numbers. Another of their properties that is analogous to

$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{a} \wedge \mathfrak{b}$	$\mathfrak{a} \wedge \mathfrak{c}$	$\mathfrak{b} \vee \mathfrak{c}$	$\mathfrak{a} \wedge (\mathfrak{b} \vee \mathfrak{c})$	$(\mathfrak{a} \wedge \mathfrak{b}) \vee (\mathfrak{a} \wedge \mathfrak{c})$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	F	F	F
F	T	T	F	F	T	F	F
F	T	F	F	F	T	F	F
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	F

Table 2.12: Truth table establishing $(\mathfrak{a} \wedge (\mathfrak{b} \vee \mathfrak{c})) \Leftrightarrow ((\mathfrak{a} \wedge \mathfrak{b}) \vee (\mathfrak{a} \wedge \mathfrak{c}))$.

that of addition and multiplication of numbers is that they have *identity (neutral) elements*. As the following proposition shows, every contradiction is an identity element of $\vee$ and every tautology is an identity element of $\wedge$.

Proposition 2.6.1 *Let $\mathfrak{a}$, $\mathfrak{c}$, and $\mathfrak{t}$ be respectively a statement, a contradiction, and a tautology. Then*

$$(\mathfrak{a} \vee \mathfrak{c}) \Leftrightarrow \mathfrak{a}, \qquad (\mathfrak{a} \wedge \mathfrak{t}) \Leftrightarrow \mathfrak{a}. \tag{2.23}$$

Proof: According to Table 2.3, because the truth value of $\mathfrak{c}$ and $\mathfrak{t}$ are respectively "F" and "T," $\mathfrak{a} \vee \mathfrak{c}$ and $\mathfrak{a} \wedge \mathfrak{t}$ have the same truth value as $\mathfrak{a}$. This implies (2.23). ■

The following proposition reveals a property of $\vee$ and $\wedge$ that is not shared with the usual addition and multiplication of numbers.

Proposition 2.6.2 *Let $\mathfrak{a}$ be a statement. Then*

$$(\mathfrak{a} \vee \mathfrak{a}) \Leftrightarrow \mathfrak{a}, \qquad (\mathfrak{a} \wedge \mathfrak{a}) \Leftrightarrow \mathfrak{a}. \tag{2.24}$$

Proof: This follows from Table 2.3. ■

Having obtained the basic properties of the operations $\neg, \wedge, \vee, \Rightarrow$ and $\Leftrightarrow$, we can perform calculations involving compound statements. This is called ***Propositional Calculus***, mainly pioneered by Gottlob Frege (1848-1925), Alfred Whitehead (1861-1947), Bertrand Russell (1872-1970), and David Hilbert (1862-1943). We have already employed it in the proof of part (b) of Theorem 2.6.1 and proof of Corollary 2.6.1. The following are other simple applications of Propositional Calculus.

Exercise 2.6.1 Let $\mathfrak{a}$ and $\mathfrak{b}$ be statements. Show that

$$\neg(\mathfrak{a} \Rightarrow \mathfrak{b}) \Leftrightarrow (\mathfrak{a} \wedge \neg\mathfrak{b}). \tag{2.25}$$

Solution: We preform the following calculation.

$$\begin{aligned} \neg(\mathfrak{a} \Rightarrow \mathfrak{b}) &\Leftrightarrow (\neg(\neg\mathfrak{a} \vee \mathfrak{b})) && \text{(by Theorem 2.6.2)} \\ &\Leftrightarrow (\neg(\neg\mathfrak{a}) \wedge \neg\mathfrak{b}) && \text{(by Theorem 2.6.1)} \\ &\Leftrightarrow (\mathfrak{a} \wedge \neg\mathfrak{b}) && \text{(by Proposion 2.3.1).} \quad \blacksquare \end{aligned}$$

Exercise 2.6.2 Let $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ be statements. Show the logical equivalence of $(\mathfrak{a} \wedge \mathfrak{b}) \Rightarrow \mathfrak{c}$ and $\mathfrak{a} \Rightarrow (\mathfrak{b} \Rightarrow \mathfrak{c})$.
Solution: We preform the following calculation.

$$\begin{aligned} ((\mathfrak{a} \wedge \mathfrak{b}) \Rightarrow \mathfrak{c}) &\Leftrightarrow (\neg(\mathfrak{a} \wedge \mathfrak{b}) \vee \mathfrak{c}) && \text{(by Theorem 2.6.2)} \\ &\Leftrightarrow ((\neg\mathfrak{a} \vee \neg\mathfrak{b}) \vee \mathfrak{c}) && \text{(by Theorem 2.6.1)} \\ &\Leftrightarrow (\neg\mathfrak{a} \vee (\neg\mathfrak{b} \vee \mathfrak{c})) && \text{(by Theorem 2.6.3)} \\ &\Leftrightarrow (\neg\mathfrak{a} \vee (\mathfrak{b} \Rightarrow \mathfrak{c})) && \text{(by Theorem 2.6.2)} \\ &\Leftrightarrow (\mathfrak{a} \Rightarrow (\mathfrak{b} \Rightarrow \mathfrak{c})) && \text{(by Theorem 2.6.2).} \quad \blacksquare \end{aligned}$$

Exercise 2.6.3 Express $\Leftrightarrow$ in terms of $\neg$ and $\vee$.
Solution: Let $\mathfrak{a}$ and $\mathfrak{b}$ be statements. Then, we have

$$\begin{aligned} (\mathfrak{a} \Leftrightarrow \mathfrak{b}) &\Leftrightarrow ((\mathfrak{a} \Rightarrow \mathfrak{b}) \wedge (\mathfrak{b} \Rightarrow \mathfrak{a})) \\ &\Leftrightarrow (\neg\mathfrak{a} \vee \mathfrak{b}) \wedge (\neg\mathfrak{b} \vee \mathfrak{a}) \\ &\Leftrightarrow \neg(\neg(\neg\mathfrak{a} \vee \mathfrak{b}) \vee \neg(\neg\mathfrak{b} \vee \mathfrak{a})). \quad \blacksquare \end{aligned}$$

Next, we give an alternative solution of Exercise 2.5.3 that uses the methods of Propositional Calculus. First we recall the statement of this Exercise.

Exercise 2.5.3: *Let* $\mathfrak{e}, \mathfrak{f}$ *and* $\mathfrak{g}$ *be statements. Show that*

$$\mathfrak{h} := (\,((\mathfrak{e} \wedge \mathfrak{f}) \wedge (\mathfrak{e} \Rightarrow \mathfrak{g})) \;\Rightarrow\; (\mathfrak{f} \wedge \mathfrak{g})\,)$$

is a tautology.
Solution: We begin our analysis by using the identity (Theorem 2.6.1)

$$(\mathfrak{e} \Rightarrow \mathfrak{g}) \Leftrightarrow (\neg\mathfrak{e} \vee \mathfrak{g}) \tag{2.26}$$

and the distribution law of $\wedge$ over $\vee$ (Theorem 2.6.2) to establish

$$((\mathfrak{e} \wedge \mathfrak{f}) \wedge (\mathfrak{e} \Rightarrow \mathfrak{g})) \Leftrightarrow ((\mathfrak{e} \wedge \mathfrak{f}) \wedge (\neg \mathfrak{e} \vee \mathfrak{g})) \Leftrightarrow ((\mathfrak{e} \wedge \mathfrak{f}) \wedge \neg \mathfrak{e}) \vee ((\mathfrak{e} \wedge \mathfrak{f}) \wedge \mathfrak{g})). \quad (2.27)$$

In view of the associativity and commutativity of $\wedge$, we have

$$((\mathfrak{e} \wedge \mathfrak{f}) \wedge \neg \mathfrak{e}) \quad \Leftrightarrow \quad (\mathfrak{f} \wedge (\mathfrak{e} \wedge \neg \mathfrak{e})), \quad (2.28)$$
$$((\mathfrak{e} \wedge \mathfrak{f}) \wedge \mathfrak{g})) \quad \Leftrightarrow \quad (\mathfrak{e} \wedge (\mathfrak{f} \wedge \mathfrak{g})). \quad (2.29)$$

Because $\mathfrak{e} \wedge \neg \mathfrak{e}$ is a contradiction, according to (2.28), so is $((\mathfrak{e} \wedge \mathfrak{f}) \wedge \neg \mathfrak{e})$. This together with (2.27) and (2.29) imply

$$((\mathfrak{e} \wedge \mathfrak{f}) \wedge (\mathfrak{e} \Rightarrow \mathfrak{g})) \Leftrightarrow ((\mathfrak{e} \wedge \mathfrak{f}) \wedge \mathfrak{g})) \Leftrightarrow (\mathfrak{e} \wedge (\mathfrak{f} \wedge \mathfrak{g})). \quad (2.30)$$

Next, consider

$$(\mathfrak{e} \wedge (\mathfrak{f} \wedge \mathfrak{g})) \Rightarrow (\mathfrak{f} \wedge \mathfrak{g}). \quad (2.31)$$

In order for this implication to be false its hypothesis $\mathfrak{e} \wedge (\mathfrak{f} \wedge \mathfrak{g})$ must be true whilst its conclusion $\mathfrak{f} \wedge \mathfrak{g}$ must be false. But this is not possible, because if $\mathfrak{f} \wedge \mathfrak{g}$ is false then so is $\mathfrak{e} \wedge (\mathfrak{f} \wedge \mathfrak{g})$. This shows that (2.31) is always true; it is a tautology. In view of (2.30), (2.31) is logically equivalent to $\mathfrak{h}$. Therefore, $\mathfrak{h}$ is also a tautology. ■

Comments and Suggestions for Further Reading

In this chapter we outlined the basic ingredients upon which we will base our logical reasoning in the rest of this book. In particular, we will use our knowledge of propositional calculus in Chapter 3 to clarify the logical underpinning of various proof methods that we will employ in our study of sets, relations, functions, etc. in Chapters 4-8.

Similar accounts of elementary logic can be found in almost every introductory textbook on abstract mathematics. Some examples are

1. R. Garnier and J. Taylor, *100% Mathematical Proof*, John Wiley & Sons, Chichester, West Sussex, 1996.

2. G. Chartrand, A. D. Polimeni, and P. Zhang, *Mathematical Proofs*, Addison Wesley, Boston, 2008.

3. C. Schumacher, *Chapter Zero*, Addison Wesley, New York, 2001.

More advanced treatment of the subject are given in

4. S. N. Burris, *Logic for Mathematics and Computer Science*, Prentice Hall, Upper Saddle River, New Jersey, 1998.

5. S. Hedman, *A First Course in Logic*, Oxford University Press, Oxford, 2004.

The methods of Logic have important applications in computer science and electrical engineering. For a readable discussion of some of these applications, see

6. A. Nerode and R. A. Shore, *Logic for Applications*, Springer-Verlag, New York, 1993.

Problems

Problem 2.1 Specify the truth value of the following statements and determine their negation.

$$\mathfrak{a}_1 := (\forall x \in \mathbb{N},\ \exists y \in \mathbb{N},\ x < y)$$

$$\mathfrak{a}_2 := (\exists y \in \mathbb{N},\ \forall x \in \mathbb{N},\ x < y)$$

$$\mathfrak{a}_3 := (\exists x \in \mathbb{N},\ \forall y \in \mathbb{N},\ x < y)$$

$$\mathfrak{a}_4 := (\forall y \in \mathbb{N},\ \exists x \in \mathbb{N},\ x < y)$$

$$\mathfrak{a}_5 := (\exists x \in \mathbb{N},\ \exists y \in \mathbb{N},\ x < y)$$

$$\mathfrak{a}_6 := (\forall x \in \mathbb{N},\ \forall y \in \mathbb{N},\ x < y)$$

Problem 2.2 Let $\mathfrak{p}(x)$ be a predicate depending on a variable x that takes values in a set A. Express the negation of the statement "$\exists! x \in A, \mathfrak{p}(x)$" using mathematical symbols.

Problem 2.3 Obtain a solution of Exercise 2.5.3 and a proof of part (b) of Theorem 2.6.1 by constructing the relevant truth tables.

Problem 2.4 Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be statements. Prove that $(\mathfrak{a} \Leftrightarrow \mathfrak{b}) \Leftrightarrow \mathfrak{c}$ is logically equivalent to $\mathfrak{a} \Leftrightarrow (\mathfrak{b} \Leftrightarrow \mathfrak{c})$, i.e., $\Leftrightarrow$ is an associative operation.

Problem 2.5 Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be statements. Determine if the following compound statements are tautologies.

$$\mathfrak{d}_1 := (\mathfrak{a} \Leftrightarrow (\mathfrak{b} \wedge \mathfrak{c})) \Leftrightarrow ((\mathfrak{a} \Leftrightarrow \mathfrak{b}) \wedge (\mathfrak{a} \Leftrightarrow \mathfrak{c}))$$

$$\mathfrak{d}_2 := (\mathfrak{a} \Leftrightarrow (\mathfrak{b} \vee \mathfrak{c})) \Leftrightarrow ((\mathfrak{a} \Leftrightarrow \mathfrak{b}) \vee (\mathfrak{a} \Leftrightarrow \mathfrak{c}))$$

$$\mathfrak{d}_3 := (\mathfrak{a} \Leftrightarrow (\mathfrak{b} \Rightarrow \mathfrak{c})) \Leftrightarrow ((\mathfrak{a} \Leftrightarrow \mathfrak{b}) \Rightarrow (\mathfrak{a} \Leftrightarrow \mathfrak{c}))$$

Problem 2.6 Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{a}', \mathfrak{b}'$ be statements such that $\mathfrak{a} \Leftrightarrow \mathfrak{a}'$ and $\mathfrak{b} \Leftrightarrow \mathfrak{b}'$. Prove that the following compound statements are tautologies.

$$\mathfrak{c}_1 := ((\mathfrak{a} \wedge \mathfrak{b}) \Leftrightarrow (\mathfrak{a}' \wedge \mathfrak{b}')).$$

$$\mathfrak{c}_2 := ((\mathfrak{a} \vee \mathfrak{b}) \Leftrightarrow (\mathfrak{a}' \vee \mathfrak{b}')).$$

$$\mathfrak{c}_3 := ((\mathfrak{a} \Rightarrow \mathfrak{b}) \Leftrightarrow (\mathfrak{a}' \Rightarrow \mathfrak{b}')).$$

Problem 2.7 Let $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ be statements, $\mathfrak{d} := (\neg\mathfrak{a} \Rightarrow (\mathfrak{b} \Rightarrow \mathfrak{c}))$, and $\mathfrak{e} := (\neg(\mathfrak{a} \Rightarrow \mathfrak{b}) \Rightarrow \mathfrak{c})$. Determine whether $\mathfrak{d} \Rightarrow \mathfrak{e}$ is a tautology?

Problem 2.8 Let $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ be statements, $\mathfrak{d} := ((\mathfrak{b} \Rightarrow \mathfrak{a}) \Rightarrow (\mathfrak{b} \wedge \mathfrak{c}))$, and $\mathfrak{e} := (\mathfrak{b} \wedge (\mathfrak{a} \Rightarrow \mathfrak{c}))$. Show the logical equivalence of $\mathfrak{d}$ and $\mathfrak{e}$ by

(a) constructing the corresponding truth table;

(b) using the methods of propositional calculus.

Problem 2.9 Repeat Problem 2.8 for $\mathfrak{d} := (\mathfrak{a} \wedge (\mathfrak{b} \Rightarrow \neg\mathfrak{a}))$ and $\mathfrak{e} := (\neg(\mathfrak{a} \Rightarrow \mathfrak{b}))$.

Problem 2.10 Let $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ be statements. For each of the following statements find a logically equivalent statement that only involves $\neg$ and $\vee$.

$$\mathfrak{d}_1 := (\mathfrak{a} \Rightarrow \neg\mathfrak{a})$$

$$\mathfrak{d}_2 := (\mathfrak{a} \Rightarrow (\mathfrak{b} \Rightarrow \mathfrak{c}))$$

$$\mathfrak{d}_3 := ((\mathfrak{a} \Rightarrow \mathfrak{b}) \wedge (\mathfrak{b} \Rightarrow \mathfrak{c}))$$

$$\mathfrak{d}_4 := (((\mathfrak{a} \Rightarrow \mathfrak{b}) \wedge (\mathfrak{b} \Rightarrow \mathfrak{c})) \Rightarrow (\mathfrak{a} \Rightarrow \mathfrak{c}))$$

$$\mathfrak{d}_5 := ((\mathfrak{a} \Leftrightarrow \mathfrak{b}) \Rightarrow \mathfrak{c})$$

Problem 2.11 Let $\mathfrak{a}_1, \mathfrak{a}_2$ and $\mathfrak{a}_3$ be statements. Prove the logical equivalence of the following compound statements.

$$\mathfrak{b}_1 := (\mathfrak{a}_1 \wedge \mathfrak{a}_2 \wedge \mathfrak{a}_3)$$

$$\mathfrak{b}_2 := (\mathfrak{a}_1 \wedge (\mathfrak{a}_1 \Rightarrow \mathfrak{a}_2 \Rightarrow \mathfrak{a}_3))$$

$$\mathfrak{b}_3 := (\mathfrak{a}_1 \wedge (\forall m \in \{1,2\}, (\mathfrak{a}_m \Rightarrow \mathfrak{a}_{m+1})))$$

Here "$\forall m \in \{1,2\}$" means "for both values 1 and 2 of m."