Solutions

Math 103: Midterm Exam # 1

Fall 2018

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 2 hours.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- No questions about the content of this exam will be answered.
- (Optional) Grade your own work out of 100. Record your estimated grade here:

Estimated Grade:	

If your expected grade and actual grade will turn out to differ by 9 points or less, you will be given the highest of the two.

To be filled by the grader:

Actual Grade:	
Adjusted Grade:	

Problem 1. (10 points) Let $\mathfrak{a}$ and $\mathfrak{b}$ be statements. Find a compound statement involving only $\mathfrak{a}$, $\mathfrak{b}$, $\neg$, and $\wedge$ that is logically equivalent to $\mathfrak{a} \Leftrightarrow \mathfrak{b}$. Justify your response by producing a relevant truth table and comparing the truth value of your response and $\mathfrak{a} \Leftrightarrow \mathfrak{b}$.

$$(a \Leftrightarrow b) \Leftrightarrow ((a \Rightarrow b) \wedge (b \Rightarrow a))$$
$$\Leftrightarrow ((\neg a \vee b) \wedge (\neg b \vee a))$$
$$\Leftrightarrow \underbrace{(\neg(a \wedge \neg b)) \wedge (\neg(b \wedge \neg a))}_{c}$$

a	b	$\neg a$	$\neg b$	$a \wedge \neg b$	$\neg(a \wedge \neg b)$	$b \wedge \neg a$	$\neg(b \wedge \neg a)$	c	$a \Leftrightarrow b$
T	T	F	F	F	T	F	T	T	T
T	F	F	T	T	F	F	T	F	F
F	T	T	F	F	T	T	F	F	F
F	F	T	T	F	T	F	T	T	T

identical.

Problem 2. (10 points) Let a and b be nonzero real numbers. Prove that $\sqrt{a^2+b^2} \neq (a^3+b^3)^{1/3}$.

Suppose by contradiction that $\sqrt{a^2+b^2} = (a^3+b^3)^{1/3}$

$\Rightarrow (a^2+b^2)^3 = (a^3+b^3)^2$

$\Rightarrow a^6 + 3a^4b^2 + 3a^2b^4 + b^6 = a^6 + 2a^3b^3 + b^6$

$\Rightarrow 3a^2b^2(a^2+b^2) - 2a^2b^2(ab) = 0$

$\Rightarrow a^2b^2(3a^2+3b^2-2ab = 0$

$\Rightarrow a^2b^2[2(a^2+b^2)+(a+b)^2] = 0$ Because $a \neq 0 \wedge b \neq 0$

$a^2b^2 \neq 0 \quad \hookrightarrow 2(a^2+b^2)+(a+b)^2 = 0 \Rightarrow \underbrace{a^2+b^2=0}_{a=b=0} \wedge (a+b)^2 = 0$

$\Rightarrow \sqrt{a^2+b^2} \neq (a^3+b^3)^{1/3}$ ∎

Problem 3. (15 points) Let n be a positive integer. Use induction to prove that the product of odd positive integers smaller than $2n$, equals $\frac{(2n)!}{2^n\, n!}$.

Let $\forall n \in \mathbb{Z}^+$, $\prod_{j=1}^{n}(2j-1) := 1 \times 3 \times 5 \times \cdots \times (2n-1)$

and $a_n := \left(\prod_{j=1}^{n}(2j-1) = \frac{(2n)!}{2^n\, n!}\right)$. We wish to prove that $\forall n \in \mathbb{Z}^+$, a_n is true.

1) For $n=1$: $\prod_{j=1}^{n}(2j-1) = 2-1 = 1$ & $\frac{(2n)!}{2^n n!} = \frac{2!}{2 \times 1!} = \frac{2}{2} = 1$

So a_1 holds.

2) Suppose $\exists m \in \mathbb{Z}^+$, a_m holds, i.e., $\prod_{j=1}^{m}(2j-1) = \frac{(2m)!}{2^m m!}$

3) Prove a_{m+1}:

$$\prod_{j=1}^{m+1}(2j-1) = \left[\prod_{j=1}^{m}(2j-1)\right][2(m+1)-1] = \frac{(2m)!}{2^m m!}\underbrace{[2(m+1)-1]}_{2m+1}$$

$$= \frac{(2m)!\,(2m+1)}{2^m m!} \quad (1)$$

$$\frac{(2(m+1))!}{2^{m+1}(m+1)!} = \frac{(2m+2)!}{(2^m m!)(2)(m+1)} = \frac{(2m)!\,(2m+1)(2m+2)}{(2^m m!)\,2(m+1)}$$

$$= \frac{(2m)!\,(2m+1)}{2^m m!} \quad (2)$$

(1) & (2) $\Rightarrow a_{m+1}$ holds. ∎

Problem 4.

4.a (2 points) Give the definition of an inductive set.

A set A is inductive, if the following hold.

1) $\phi \in A$.

2) $\forall x \in A$, $S(x) \in A$,

where $S(x) := x \cup \{x\}$.

4.b (3 points) Give the definition of the set of natural numbers $\mathbb{N}$.

Let J be an inductive set

$$\mathbb{N} := \{ n \in J \mid \text{if } K \text{ is an inductive set, } n \in K \}$$

4.c (3 points) Prove that $\mathbb{N}$ is a set. An inductive set exists by the axiom of infinity. We choose it to be the J in the above definition. Then (K is inductive $\Rightarrow n \in K$) is a predicate with variable $n \in J$ $\Rightarrow$ $\mathbb{N}$ is a set by specification axiom.

4.d (7 points) Prove that $\mathbb{N}$ is an inductive set.

Every inductive set includes ϕ as an element. $\mathbb{N}$ consists of common elements of inductive set

$\Rightarrow \phi \in \mathbb{N}$. 1

Let $n \in \mathbb{N}$, then $n \in K$ for every inductive set

$\Downarrow$

because K is inductive

$n+1 = S(n) \in K$

$\Rightarrow$ $S(n)$ belongs to every inductive set

$\Rightarrow$ $S(n) \in \mathbb{N}$. 2

1 & 2 $\Rightarrow$ $\mathbb{N}$ is inductive. ☒

Problem 5. (15 points) Let A and B be nonempty sets, $R \subseteq A \times B$, $\{B_\alpha\}_{\alpha \in \mathscr{A}}$ be a collection of subsets of B. Prove that $R^{-1}(\bigcap_{\alpha \in \mathscr{A}} B_\alpha) \subseteq \bigcap_{\alpha \in \mathscr{A}} R^{-1}(B_\alpha)$.

$$\forall a \in R^{-1}(\bigcap_{\alpha \in \mathscr{A}} B_\alpha), \quad \exists b \in \bigcap_{\alpha \in \mathscr{A}} B_\alpha, \quad aRb$$

$$\Downarrow$$

$$\forall \alpha \in \mathscr{A}, \quad \underbrace{b \in B_\alpha, \quad aRb}$$

$$\Downarrow$$

$$\forall \alpha \in A, \quad a \in R^{-1}(B_\alpha)$$

$$\Rightarrow a \in \bigcap_{\alpha \in \mathscr{A}} R^{-1}(B_\alpha)$$

$$\Rightarrow R^{-1}(\bigcap_{\alpha \in \mathscr{A}} B_\alpha) \subseteq \bigcap_{\alpha \in \mathscr{A}} R^{-1}(B_\alpha). \quad \blacksquare$$

Problem 6. Let A, B, and C be nonempty sets, $R \subseteq A \times B$ and $S \subseteq B \times S$.

6.b (2 points) Give the definition of $S \circ R$.

$$S \circ R := \{ (a,c) \in A \times C \mid \exists b \in B,\ \underbrace{a R b}_{\Updownarrow\ (a,b) \in R} \wedge \underbrace{b S c}_{\Updownarrow\ (b,c) \in S} \}$$

6.b (8 points) Show that $\mathrm{Dom}(S \circ R) = R^{-1}(\mathrm{Dom}(S))$, where "Dom" means the domain of its argument.

$\forall a \in \mathrm{Dom}(S \circ R) = (S \circ R)^{-1}(C)$, $\exists c \in C$, $(a,c) \in S \circ R$

$\Rightarrow \exists b \in B$, $(a,b) \in R \wedge (b,c) \in S$

$\Downarrow$

$b \in \mathrm{Dom}(S)$

$\Downarrow$

$a \in R^{-1}(\mathrm{Dom}(S))$

$\Rightarrow \mathrm{Dom}(S \circ R) \subseteq R^{-1}(\mathrm{Dom}(S))$ (1)

$\forall a' \in R^{-1}(\mathrm{Dom}(S))$, $\exists b' \in \mathrm{Dom}(S)$, $(a',b') \in R$

$\Downarrow$

$\exists c' \in C$, $(b',c') \in S$

$\Downarrow$

$(a',b') \in R \wedge (b',c') \in S$

$\Downarrow$

$(a',c') \in S \circ R$

$\Downarrow$

$a' \in \mathrm{Dom}(S \circ R)$

$\Rightarrow R^{-1}(\mathrm{Dom}(S)) \subseteq \mathrm{Dom}(S \circ R)$ (2)

(1) & (2) $\Rightarrow$ $\mathrm{Dom}(S \circ R) = R^{-1}(\mathrm{Dom}(S))$ $\square$

Problem 7. Let A be a nonempty set, a_1 and a_2 be elements of A, $\sim \subseteq A \times A$ be an equivalence relation, and $[a_1]$ and $[a_2]$ be respectively the equivalence classes of a_1 and a_2

7.a (6 points) Show that $a_1 \sim a_2$ if and only if $[a_1] = [a_2]$.

($\Rightarrow$) Suppose that $a_1 \sim a_2$. Then $\forall a \in [a_1]$, $a \sim a_1$ ①

So $a \sim a_1 \wedge a_1 \sim a_2 \Rightarrow a \sim a_2 \Rightarrow a \in [a_2] \Rightarrow [a_1] \subseteq [a_2]$

Similarly $\forall a' \in [a_2]$, $a' \sim a_2$ ②

$a_1 \sim a_2 \Rightarrow a_2 \sim a_1$ $\Rightarrow a' \sim a_1 \Rightarrow a' \in [a_1] \Rightarrow [a_2] \subseteq [a_1]$

① & ② $\Rightarrow$ $[a_1] = [a_2]$

($\Leftarrow$) Suppose that $[a_1] = [a_2]$.

$\underbrace{a_1 \sim a_1}_{\sim \text{ is reflexive}} \Rightarrow a_1 \in [a_1] \Rightarrow a_1 \in [a_2] \Rightarrow a_1 \sim a_2$ ∎

7.b (4 points) Show that $[a_1] \neq [a_2]$ if and only if $[a_1] \cap [a_2] = \emptyset$.

This statement is logically equivalent to

$[a_1] \cap [a_2] \neq \phi$ if and only if $[a_1] = [a_2]$.

We prove the latter.

($\Rightarrow$) Suppose that $[a_1] \cap [a_2] \neq \phi \Rightarrow \exists x \in [a_1] \cap [a_2]$

$\Rightarrow x \in [a_1] \wedge x \in [a_2] \Rightarrow x \sim a_1 \wedge x \sim a_2$

$\underbrace{x \sim a_1 \Rightarrow a_1 \sim x}_{\sim \text{ is symmetric}}$ $\xrightarrow{\sim \text{ is transitive}}$ $a_1 \sim a_2 \Rightarrow [a_1] = [a_2]$

Problem 7.a.

($\Leftarrow$) Suppose that $[a_1] = [a_2] \Rightarrow [a_1] \cap [a_2] = [a_1]$

$\underbrace{a_1 \sim a_1}_{\sim \text{ is reflexive}} \Rightarrow a_1 \in [a_1] \Rightarrow [a_1] \neq \phi \Rightarrow [a_1] \cap [a_2] \neq \phi$.

∎

Problem 8. Let A be a nonempty set, $R \subseteq A \times A$, and $S \subseteq A \times A$.

8.a (2 points) Show that if R and S are reflexive, then so is $S \circ R$.

$\forall a \in A, \quad aRa \wedge aSa \Rightarrow (a,a) \in S \circ R$

R is reflexive   S is reflexive

$\Rightarrow S \circ R$ is reflexive. ☒

8.b (5 points) Show that if R is both antisymmetric and transitive, then $R \circ R$ is antisymmetric .

$\forall x, y \in A, \quad (x,y) \in R \circ R \ ① \wedge (y,x) \in R \circ R \ ②$

① $\Rightarrow \exists u \in A, \ (x,u) \in R \wedge (u,y) \in R \Rightarrow (x,y) \in R$

② $\Rightarrow \exists v \in A, \ (y,v) \in R \wedge (v,x) \in R \Rightarrow (y,x) \in R$

because R is transitive

$\Rightarrow x = y$ ③ because R is antisymmetric

① $\wedge$ ② $\Rightarrow$ ③ so $R \circ R$ is antisymmetric. ☒

8.c (8 points) Show that if R is an equivalence relation, then $R \circ R = R$.

$\forall (x,y) \in R \circ R, \ \exists z \in A, \ (x,z) \in R \wedge (z,y) \in R$

$\Rightarrow (x,y) \in R$ because R is transitive.

$\Rightarrow R \circ R \subseteq R$ ①

$\forall (p,q) \in R, \quad q \in A \Rightarrow (q,q) \in R$ because R is reflexive

$\Downarrow$

$(p,q) \in R \circ R$

$\Rightarrow R \subseteq R \circ R$ ②

① & ② $\Rightarrow R \circ R = R$. ☒