2. Let A, B, C be nonempty sets, $X \subseteq A \times B$, and $Y \subseteq B \times C$. Prove the following statements.

$\mathfrak{a}$:= $(\text{Ran}(Y \circ X) = Y(\text{Ran}(X)))$ (15 points)

Let $c \in Ran(Y \circ X)$. Then, $\exists a \in A$ s.t. $a \, Y \circ X \, c$, which gives, $\exists b \in B$ s.t. $(aXb) \wedge (bYc)$. aXb implies $b \in Ran\,X$, and then, c is in the image of $Ran\,X$ under Y, since bYc, i.e. $c \in Y(Ran(X))$. So, $Ran(Y \circ X) \subseteq Y(Ran\,X)$

Let now $c \in Y(Ran(X))$. Then, $\exists$ $b \in Ran\,X$ s.t. bYc. $b \in Ran\,X$ gives, $\exists a \in A$ s.t aXb. So, $(aXb) \wedge (bYc)$, i.e. $a \, Y \circ X \, c$, so $c \in Ran(Y \circ X)$.

Therefore, $Ran(Y \circ X) \subseteq Y(Ran(X))$.

Finally, $Ran(Y \circ X) = Y(Ran(X))$

$\mathfrak{b}$:= $(\text{Dom}(X^{-1} \circ X) = \text{Dom}(X))$ (15 points)

Let $a \in Dom\,X$. Then, $\exists b \in B$ s.t. aXb. So, $(aXb) \wedge (bX^{-1}a)$, i.e. $a \, X^{-1} \circ X \, a$. Therefore, $a \in Dom(X^{-1} \circ X)$. So, $Dom\,X \subseteq Dom(X^{-1} \circ X)$.

Let now $a \in Dom(X^{-1} \circ X)$. So, $\exists a' \in A$ with $a \, X^{-1} \circ X \, a'$, i.e. $\exists b \in B$ s.t. $(aXb) \wedge (bX^{-1}a')$. Or, $aXb \Rightarrow a \in Dom\,X$, i.e. $Dom(X^{-1} \circ X) \subseteq Dom\,X$

Finally, $Dom\,X = Dom\,X^{-1} \circ X$.

3. Let P be the set of people, $F := \{(a, b) \in P^2 \mid a \text{ is } b\text{'s father.}\}$, and $D := \{(a, b) \in P^2 \mid a \text{ is } b\text{'s daughter.}\}$. Determine the domain of $F \circ D^{-1}$. (20 points)

$$\text{Dom } F \circ D^{-1} = \{x \in P \mid \exists y \in P,\ x\, F\circ D^{-1} y\}$$
$$= \{x \in P \mid \exists y \in P, \exists z \in P, (x D^{-1} z) \wedge (z F y)\}$$
$$= \{x \in P \mid \exists y \in P, \exists z \in P, (z D x) \wedge (z F y)\}$$
$$= \{x \in P \mid \exists y \in P, \exists z \in P, (z \text{ is } x\text{'s daughter}) \wedge (z \text{ is } y\text{'s father})\}$$

Or, (z is x's daughter) $\Rightarrow$ z is a female

(z is y's father) $\Rightarrow$ z is a male

So, no z can exist. Hence, $\text{Dom } F \circ D^{-1} = \emptyset$.

4. Find all possible equivalence relations on the set $\{0, 1, 2, 3\}$. (30 points)

$\sim_1 := \{0,1,2,3\}^2$

$\sim_2 := \{0,1,2\}^2 \cup \{3\}^2$

$\sim_3 := \{0,1,3\}^2 \cup \{2\}^2$

$\sim_4 := \{0,2,3\}^2 \cup \{1\}^2$

$\sim_5 := \{1,2,3\}^2 \cup \{0\}^2$

$\sim_6 := \{0,1\}^2 \cup \{2,3\}^2$

$\sim_7 := \{0,2\}^2 \cup \{1,3\}^2$

$\sim_8 := \{0,3\}^2 \cup \{1,2\}^2$

$\sim_9 := \{0,1\}^2 \cup \{2\}^2 \cup \{3\}^2$

$\sim_{10} := \{0,2\}^2 \cup \{1\}^2 \cup \{3\}^2$

$\sim_{11} := \{0,3\}^2 \cup \{1\}^2 \cup \{2\}^2$

$\sim_{12} := \{1,2\}^2 \cup \{0\}^2 \cup \{3\}^2$

$\sim_{13} := \{1,3\}^2 \cup \{0\}^2 \cup \{2\}^2$

$\sim_{14} := \{2,3\}^2 \cup \{0\}^2 \cup \{1\}^2$

$\sim_{15} := \{1\}^2 \cup \{2\}^2 \cup \{3\}^2 \cup \{0\}^2$

It's easy to see that the partitions corresponding to those relations cover all possible partitions of $\{0,1,2,3\}$. Therefore, since every equivalence relation is obtained from some partition in a unique way, there are no other equivalence relations.