Solutions

Math 103: Quiz # 2

Fall 2007

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 45 minutes.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)

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1.a) Explain what a trivial proof is. (5 points)

See the book.

1.b) Give the statement of the division algorithm. (5 points)

See the book.

1.c) Give the definition of a prime number. (5 points)

See the book.

1.d) Give the statement of the induction axiom. (5 points)

See the book.

2. Prove or disprove the following statements. (20 points)

$a := (\exists n \in \mathbb{N},\ n! - 1$ is a prime number)

$3 \in \mathbb{N}$ and $3! - 1 = 6 - 1 = 5$ is a prime number.

Therefore a is true. ▨

$b := (\forall n \in \mathbb{N},\ n! - 1$ is a prime number)

$1 \in \mathbb{N}$ and $1! - 1 = 1 - 1 = 0$ is not a prime

number. Therefore b is is false. ▨

3. Prove that for all $r \in \mathbb{R}^+$ and $n \in \mathbb{N}$ if $n \geq 2$, $(1+r)^n > 1 + nr$. (30 points)

Let $r \in \mathbb{R}^+$ be arbitrary and $\forall n \in \mathbb{N}$ and $n \geqslant 2$

$a_n := ((1+r)^n > 1 + nr)$

We prove a_n for all $n \geqslant 2$ by induction.

<u>Step 1</u>: For $n = 2$: $a_n = ((1+r)^2 > 1 + 2r)$

$(1+r)^2 = 1 + 2r + r^2 > 1 + 2r$ so a_n is true for $n=2$.

<u>Step 2</u>: Suppose $\exists m \in \mathbb{N}, (m \geqslant 2 \wedge a_m)$, i.e.,

$(1+r)^m > 1 + mr$.

<u>Step 3</u>: Prove $a_{m+1} := ((1+r)^{m+1} > 1 + (m+1)r)$.

$(1+r)^{m+1} = (1+r)^m (1+r)$

$r \in \mathbb{R}^+ \Rightarrow 1 + r > 0$ and $(1+r)^m > 1 + mr$ by step 2

Also $1 + mr > 0$

$$(1+r)^{m+1} = (1+r)^m (1+r) > (1+mr)(1+r)$$

$\Rightarrow (1+r)^{m+1} > 1 + (m+1)r + mr^2 > 1 + (m+1)r$

↳ because $mr^2 > 0$

$\Rightarrow a_{m+1}$. ▨

4. Prove that for all $m, n \in \mathbb{Z}$ the number $m + n\sqrt{2}$ is a rational number if and only if $n = 0$. (30 points)

$(\Rightarrow)$ Suppose $m + n\sqrt{2} \in \mathbb{Q}$. Let $r := m + n\sqrt{2}$

By contradiction suppose $n \neq 0 \Rightarrow$

$\sqrt{2} = \frac{r-m}{n} \in \mathbb{Q}$. This contradicts $\sqrt{2} \notin \mathbb{Q}$

that we proved in class. Therefore, by $*$

$n = 0$.

$(\Leftarrow)$ Suppose $n = 0 \Rightarrow m + n\sqrt{2} = m \in \mathbb{Z}$

Because $\mathbb{Z} \subseteq \mathbb{Q}$, $m \in \mathbb{Q} \Rightarrow m + n\sqrt{2} \in \mathbb{Q}$. $\blacksquare$