Solutions

Math 103: Quiz # 1

Fall 2007

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 50 minutes.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)

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1. Give the definition of the following.

1.a) the negation of a statement (5 points): Let a be a statement. The $\neg a := (a \text{ is false})$ is called the negation of a.

1.b) equality of two predicates (5 points): Let p and q be predicates. p is said to be equal to q if p and q have the same variables and for each value of these variables p and q are equal statements, i.e., they have the same meaning.

1.c) a tautology (5 points): A compound statement is called a tautology if its truth value is true regardless of the truth values of its constituent statements.

1.d) the symbol "$a \Rightarrow b \Rightarrow c$" where a, b, c are statements (5 points):

$$(a \Rightarrow b \Rightarrow c) := ((a \Rightarrow b) \wedge (b \Rightarrow c)).$$

2. Find the negation of the following statement. (15 points)

$$\mathfrak{a} := (\forall x \in \mathbb{R},\ \exists! n \in \mathbb{N},\ \forall m \in \mathbb{Z}, (xn + m^3 < 1) \vee (xm + n^3 \geq -1))$$

let $p(x) := (\exists ! n \in \mathbb{N}, \forall m \in \mathbb{Z}, (xn + m^3 < 1) \vee (xm + n^3 \geq -1))$

and $q(x,n) := (\forall m \in \mathbb{Z}, (xn + m^3 < 1) \vee (xm + n^3 \geq -1))$

So that $a = (\forall x \in \mathbb{R}, p(x))$ and

$p(x) = (\exists ! n \in \mathbb{N}, q(x,n))$

$\Rightarrow \quad \neg a = (\exists x \in \mathbb{R}, \neg p(x))$

$\neg p(x) = (\forall n \in \mathbb{N}, \neg q(x,n)) \vee (\exists n_1 \in \mathbb{N}, \exists n_2 \in \mathbb{N}, (n_1 \neq n_2)$
$\wedge\ q(x,n_1) \wedge q(x,n_2))$

$\neg q(x,n) = (\exists m \in \mathbb{Z}, \neg (xn + m^3 < 1) \vee (xm + n^3 \geq -1)))$
$= \exists m \in \mathbb{Z}, (xn + m^3 \geq 1) \wedge (xm + n^3 < -1)$

So $\neg a = (\exists x \in \mathbb{R}, (\forall n \in \mathbb{N}, \exists m \in \mathbb{Z}, (xn + m^3 \geq 1) \wedge (xm + n^3 < -1))$
$\vee (\exists n_1 \in \mathbb{N}, \exists n_2 \in \mathbb{N}, (n_1 \neq n_2) \wedge$
$(\forall m_1 \in \mathbb{Z}, (xn_1 + m_1^3 < 1) \vee (xm_1 + n_1^3 \geq -1)) \wedge$
$(\forall m_2 \in \mathbb{Z}, (xn_2 + m_2^3 < 1) \vee (xm_2 + n_2^3 \geq -1))))$ $\boxtimes$

3. Let $\mathfrak{a}$ be a statement and $\mathfrak{c} := ((\mathfrak{a} \Rightarrow \neg\mathfrak{a}) \wedge (\neg\mathfrak{a} \Rightarrow \mathfrak{a}))$. Prove that $\mathfrak{c}$ is a contradiction. (15 points)

a	$\neg a$	$a \Rightarrow \neg a$	$\neg a \Rightarrow a$	c
T	F	F	T	F
F	T	T	F	F

So c is always false. $\boxtimes$

4. Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be statements and $\mathfrak{t} := (((\mathfrak{a} \wedge \mathfrak{b}) \wedge (\mathfrak{a} \Rightarrow \mathfrak{c})) \Rightarrow (\mathfrak{b} \wedge \mathfrak{c}))$. Show without using a truth table that $\mathfrak{t}$ is a tautology. (25 points)

See p. 20, Exercise 2.5.3.

5. Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ be statements, $\mathfrak{e} := (\mathfrak{a} \wedge \mathfrak{b} \wedge \mathfrak{c} \wedge \mathfrak{d})$, and $\mathfrak{g} := (\mathfrak{a} \wedge (\mathfrak{a} \Rightarrow \mathfrak{b} \Rightarrow \mathfrak{c} \Rightarrow \mathfrak{d}))$. Prove that $\mathfrak{e} \Leftrightarrow \mathfrak{g}$. (25 points)

$(e \Leftrightarrow g) \Leftrightarrow (e \Rightarrow g) \wedge (g \Rightarrow e)$

We prove that $e \Rightarrow g$ and $g \Rightarrow e$ are both true.

- To prove $e \Rightarrow g$: Suppose e is true $\Rightarrow$ a, b, c, d are all true $\Rightarrow$ $a \Rightarrow b \wedge b \Rightarrow c \wedge c \Rightarrow d$ are true because both their hypothesis and conclusion are true.
$\Rightarrow$ g is true because a is true and
$(a \Rightarrow b \Rightarrow c \Rightarrow d) := ((a \Rightarrow b) \wedge (b \Rightarrow c) \wedge (c \Rightarrow d))$

- To prove $g \Rightarrow e$: Suppose g is true $\Rightarrow$ a is true and $a \Rightarrow b \Rightarrow c \Rightarrow d$ is true
$\Rightarrow$ $(a \Rightarrow b) \wedge (b \Rightarrow c) \wedge (c \Rightarrow d)$ is true.

$\Rightarrow$
  - $a \Rightarrow b$ is true with a being to true this implies that b is true.
  - $b \Rightarrow c$ is true but b is true $\Rightarrow$ c is true
  - $c \Rightarrow d$ is true but c is true $\Rightarrow$ d is true

So a, b, c, d are all true $\Rightarrow$ e is true. $\square$