Complete Induction:

In class we have proved the following theorem using the Induction Axiom.

Theorem 1 (Principle of Mathematical Induction): Let $\forall n \in \mathbb{Z}^+$, p_n be a statement satisfying the following two conditions.

(1) p_1 is true;

(2) $\forall k \in \mathbb{Z}^+$, $p_k \Rightarrow p_{k+1}$.

Then p_n is true $\forall n \in \mathbb{Z}^+$.

The aim of this note is to use this theorem to prove the following.

Theorem 2 (Principle of Complete Induction): Let $\forall n \in \mathbb{Z}^+$, p_n be a statement satisfying the following two conditions.

(1) p_1 is true;

(2') $\forall k \in \mathbb{Z}^+$, $(p_1 \wedge p_2 \wedge \cdots \wedge p_k) \Rightarrow p_{k+1}$.

Then p_n is true $\forall n \in \mathbb{Z}^+$.

Proof: It is sufficient to show that condition (2) of Theorem 1 holds. According to (2'), $\forall k \in \mathbb{Z}^+$, $(p_1 \wedge p_2 \wedge \cdots \wedge p_{k-1} \wedge p_k) \Rightarrow p_{k+1}$. Let $q_k := p_1 \wedge p_2 \wedge \cdots \wedge p_{k-1}$. Then in view of the identities:

$$(a \Rightarrow b) \Leftrightarrow (\sim a \vee b), \qquad (\sim (a \wedge b)) \Leftrightarrow (\sim a \vee \sim b),$$

we have

$$\begin{aligned} ((p_1 \wedge p_2 \wedge \cdots \wedge p_{k-1} \wedge p_k) \Rightarrow p_{k+1}) &\Leftrightarrow ((q_k \wedge p_k) \Rightarrow p_{k+1}) \\ &\Leftrightarrow ((\sim (q_k \wedge p_k)) \vee p_{k+1}) \\ &\Leftrightarrow ((\sim q_k \vee \sim p_k) \vee p_{k+1}) \\ &\Leftrightarrow (\sim q_k \vee (\sim p_k \vee p_{k+1})) \\ &\Leftrightarrow \sim q_k \vee (p_k \Rightarrow p_{k+1}) \qquad (\star) \end{aligned}$$

Therefore, according to (2'), $\sim q_k \vee (p_k \Rightarrow p_{k+1})$ is true. We will show that this implies that $p_k \Rightarrow p_{k+1}$ is true by proving that $\sim q_k$ is false, i.e., q_k is true. We do this using both (1) and (2').

Assume by contradiction that q_k is false. Because $q_k := p_1 \wedge p_2 \wedge \cdots \wedge p_{k-1}$ this implies that there is $j_1 < k$ such that p_{j_1} is false. Now because $j_1 \in \mathbb{Z}^+$ according to (2') we have $(p_1 \wedge \cdots \wedge p_{j_1-1}) \Rightarrow p_{j_1}$. Hence p_{j_1} is false only if $p_1 \wedge \cdots \wedge p_{j_1-1}$ is false. This in turn implies that there is $j_2 < j_1$ such that p_{j_2} is false. Again $j_2 \in \mathbb{Z}^+$ and (2') implies $(p_1 \wedge \cdots \wedge p_{j_2-1}) \Rightarrow p_{j_2}$, so $p_1 \wedge \cdots \wedge p_{j_2-1}$ must be false. This means that there is $j_3 < j_2$ such that p_{j_3} is false. If we continue this argument ℓ times we find $j_\ell < j_{\ell-1} < \cdots < j_2 < j_1 < k$ such that p_{j_ℓ} is false. Therefore, at most for $\ell = k-1$, we find that p_1 must be false which contradicts (1). Hence by contradiction q_k is true, and $\sim q_k$ is false. This together with the fact (established above) that $\sim q_k \vee (p_k \Rightarrow p_{k+1})$ is true implies that $p_k \Rightarrow p_{k+1}$ must be true. Hence (2) holds. Because (1) also holds by the hypothesis of Theorem 2, both the conditions of Theorem 1 are satisfied. Hence p_n is true for all $n \in \mathbb{Z}^+$. $\square$

Remark: According to the identity $(\star)$, $p_k \Rightarrow p_{k+1}$ implies $(p_1 \wedge p_2 \wedge \cdots \wedge p_{k-1} \wedge p_k) \Rightarrow p_{k+1}$. Hence if condition (2) of Theorem 1 holds, so does condition (2') of Theorem 2. This means that not only Theorem 1 implies Theorem 2, but the opposite is also true, i.e., these two theorems are equivalent.